Fundamentals of Quantum Optics

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1 HAMILTONIAN FORMULATION OF THE ELECTROMAGNETIC FIELD

First of all let us write down the Maxwell equations in vacuum:

\[ \nabla \cdot \mathbf{E}(r,t) = 0, \]
\[ \nabla \times \mathbf{E}(r,t) = -\frac{\partial \mathbf{B}(r,t)}{\partial t}, \]
\[ \nabla \cdot \mathbf{B}(r,t) = 0, \]
\[ \nabla \times \mathbf{B}(r,t) = \frac{1}{c^2} \frac{\partial \mathbf{E}(r,t)}{\partial t}. \]

Eq. (1.1c) is also satisfied if \( \mathbf{B}(r,t) \) is derived from a potential \( \mathbf{A}(r,t) \), i.e.,

\[ \mathbf{B}(r,t) = \nabla \times \mathbf{A}(r,t). \]

Substituting Eq. (1.2) into Eq. (1.1c) leads to

\[ \nabla \cdot (\nabla \times \mathbf{A}) = 0. \]

And substituting Eq. (1.2) into Eq. (1.1b) we get

\[ \nabla \times \mathbf{E} + \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = 0. \]

In Eq. (1.4) we can factor out the curl operator to obtain

\[ \nabla \times \left[ \mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} \right] = 0. \]

Furthermore, if we define

\[ \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi, \]

we have that

\[ \nabla \times \left[ \mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} \right] = -\nabla \times \nabla \phi = 0. \]

So now we have the electric and magnetic field, given as a function of the vector potential \( \mathbf{A} \) and the scalar potential \( \phi \). The evolution equations for the potentials we can get by using eq. (1.1a) and eq. (1.1d).

Equation (1.1a) leads to

\[ \nabla \cdot \left( -\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi \right) = 0 \]
\[ \Rightarrow \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) + \nabla^2 \phi = 0 \]
and from eq.\((1.1d)\), using the vector identity \(\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\), we get:

\[
\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial}{\partial t} \left( -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right)
\]

\[\Rightarrow \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c^2} \frac{\partial \nabla \phi}{\partial t} \quad (1.9)\]

**Note:**

Eq.\((1.8)\) and eq.\((1.9)\) Maxwell’s equations for the field potentials. However \(\mathbf{A}\) and \(\phi\) have no physical meaning. In order to show what is meant by this statement let us have a look at the following transformations

\[
\mathbf{A} = \mathbf{A}' - \nabla \Lambda, \quad \text{(1.10a)}
\]

\[
\phi = \phi' + \frac{\partial \Lambda}{\partial t}, \quad \text{(1.10b)}
\]

where \(\Lambda\) is an arbitrary (but well behaved) scalar field, which is called gauge field.

We now calculate the fields \(\mathbf{E}'\) and \(\mathbf{B}'\) generated by \(\mathbf{A}'\) and \(\phi'\). For the magnetic field \(\mathbf{B}'\) we obtain following result:

\[
\mathbf{B}' = \nabla \times \mathbf{A}'
\]

\[
= \nabla \times \mathbf{A} + \nabla \times (\nabla \Lambda)
\]

\[
= \nabla \times \mathbf{A} = \mathbf{B}, \quad (1.11)
\]

while for the electric field \(\mathbf{E}'\) we have:

\[
\mathbf{E}' = -\frac{\partial \mathbf{A}'}{\partial t} - \nabla \phi'
\]

\[
= -\frac{\partial \mathbf{A}}{\partial t} \nabla \Lambda - \nabla \phi + \frac{\partial \nabla \Lambda}{\partial t}
\]

\[
= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi = \mathbf{E}. \quad (1.12)
\]

So we conclude:

- Different field potentials generate the same \(\mathbf{E}\) and \(\mathbf{B}\) (gauge transformations)

- While \(\mathbf{E}\) and \(\mathbf{B}\) are manifestly gauge invariant, the field potentials are not. It is therefore important when dealing with field potentials to fix the gauge by fixing the choice of \(\Lambda\).

### 1.1 Coulomb Gauge

Assume to consider the condition

\[
\nabla \cdot \mathbf{A} = 0 \quad (1.13)
\]
to be valid. Therefore using this gauge we obtain for the scalar potential from Eq. (1.8) that

$$\nabla^2 \phi = 0,$$

(1.14)

from which it follows that $\phi = 0$. This result is in accordance with the fact that the scalar potential is associated to the charge distribution (and, ultimately, to the electrostatic field) and therefore since in vacuum there are no charges, the scalar potential is zero. Substituting this result into Eq. (1.9) then gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = 0.$$

(1.15)

From both, Eq. (1.14) and Eq. (1.15) we can see that in the Coulomb (radiation) gauge in vacuum, $\mathbf{E}$ and $\mathbf{B}$ are completely determined by the vector potential solely, i.e.,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

(1.16a)

and

$$\mathbf{B} = \nabla \times \mathbf{A}.$$  

(1.16b)

**What about $\Lambda$?**

Once the gauge condition is fixed, it must be respected by all $(\mathbf{A}, \phi)$. Therefore:

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A} - \nabla \cdot (\nabla \Lambda) = 0 \Rightarrow \nabla^2 \Lambda = 0$$

(1.17)

and

$$\phi = \phi' + \frac{\partial \Lambda}{\partial t} = 0 \Rightarrow \frac{\partial \Lambda}{\partial t} = 0$$

(1.18)

This means, that $\Lambda$ is a harmonic function, i.e., it is a solution of the wave equation

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Lambda = 0.$$  

(1.19)

### 1.2 Field Energy Density

Let us consider the electromagnetic field Hamiltonian:

$$H = \frac{1}{2} \int dV \left[ \varepsilon_0 |\mathbf{E}(\mathbf{r}, t)|^2 + \frac{1}{\mu_0} |\mathbf{B}(\mathbf{r}, t)|^2 \right].$$

(1.20)

Our goal is now to rewrite Eq. (1.20) in terms of the vector potential. To do that, however, we make sure to follow the following prescriptions, which will turn out to be useful for quantization.

a) **Finite Volume** $V = L^3 \Rightarrow \int dV \rightarrow \sum_k$

This is done because by introducing a finite volume we can imagine to write the
field as a superposition of cavity modes, that constitute a **finite** set of discrete basis vectors;

b) **Allowed Modes**

\[ \mathbf{k} = \frac{2\pi}{L} (n_x \mathbf{\hat{x}} + n_y \mathbf{\hat{y}} + n_z \mathbf{\hat{z}}) \quad (1.21) \]

These are the k-vectors of the modes sustained by the fictitious cavity introduced in a);

c) **Polarization**

The electromagnetic field can be characterized by 2 orthogonal states of polarization in the transverse plane (e.g., \{\(|H\rangle, |V\rangle\}, \{\(|L\rangle, |R\rangle\}\}, \{\(|TE\rangle, |TM\rangle\})...).

For each mode we then have two possible polarization states, namely \(\mathbf{\hat{e}}_{k\lambda}\), with \(\lambda = 1, 2\) and \(\mathbf{\hat{e}}_{k\lambda} \cdot \mathbf{\hat{e}}_{k\lambda'} = \delta_{\lambda\lambda'}\). The vector potential then can be written in the following way:

\[ \mathbf{A} (\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{\lambda=1}^{2} \mathbf{\hat{e}}_{k\lambda} A_{k\lambda} (\mathbf{r}, t) \quad (1.22) \]

Moreover, \(\mathbf{A}\) must obey Coulomb gauge condition, i.e., \(\nabla \cdot \mathbf{A} = 0\). This leads to

\[ \mathbf{k} \cdot \mathbf{\hat{e}}_{k\lambda} = 0, \quad (1.23) \]

which states that the vector potential (and therefore \(\mathbf{E}\) and \(\mathbf{B}\)) does not possess components along the propagation direction \(\mathbf{k}\). In other words, \(\mathbf{A}\) is a purely transverse field. Another condition that must be fulfilled, is that \(\mathbf{A}\) must obey the wave equation

\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \quad (1.24) \]

Inserting Eq.(1.22) leads to

\[ \sum_{\mathbf{k}} \sum_{\lambda=1}^{2} \mathbf{\hat{e}}_{k\lambda} \left[ \nabla^2 A_{k\lambda} (\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 A_{k\lambda} (\mathbf{r}, t)}{\partial t^2} \right] = 0. \quad (1.25) \]

Let us now assume that

\[ A_{k\lambda} (\mathbf{r}, t) = A_{k\lambda} (t) e^{ik\cdot\mathbf{r}} + A_{k\lambda}^* (t) e^{-ik\cdot\mathbf{r}}, \quad (1.26) \]

thus we have:

\[ \nabla^2 A_{k\lambda} (\mathbf{r}, t) = -\mathbf{k} \cdot \mathbf{k} \left[ A_{k\lambda} (t) e^{ik\cdot\mathbf{r}} + A_{k\lambda}^* (t) e^{-ik\cdot\mathbf{r}} \right]. \quad (1.27) \]

Defining \(\omega_k = ck\) we have from Eq.(1.25)

\[ \frac{\partial^2 A_{k\lambda} (t)}{\partial t^2} + \omega_k^2 A_{k\lambda} (t) = 0 \quad (1.28) \]
and the same expression for the complex conjugate. The solution to Eq. (1.28) is then
\[ A_{k\lambda}(t) = A_{k\lambda}e^{-i\omega_k t}, \]  
(1.29)
and for the complex conjugate
\[ A^*_{k\lambda}(t) = A^*_{k\lambda}e^{i\omega_k t}. \]  
(1.30)

Using this result and Eqs. (1.16), the electric and magnetic fields can be then written as follows:
\[ E(r,t) = \sum \sum i\omega_k \hat{e}_{k\lambda} \left[ A_{k\lambda}e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} - A^*_{k\lambda}e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \right], \]  
(1.31)
and
\[ B(r,t) = \sum \sum i (\mathbf{k} \times \hat{e}_{k\lambda}) \left[ A_{k\lambda}e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} - A^*_{k\lambda}e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \right]. \]  
(1.32)

Now we can calculate the electromagnetic field Hamiltonian \( H \) which is given by Eq. (1.20) and which we will write in following form:
\[ H = \frac{1}{2} \int dV \left[ \varepsilon_0 |E(r,t)|^2 + \frac{1}{\mu_0} |B(r,t)|^2 \right] = H_E + H_B, \]  
(1.33)
where \( H_E \) and \( H_B \) are the Hamiltonians of the electric and magnetic field, respectively. In the following we will calculate explicitly the electric field Hamiltonian, using the vector Potential \( \mathbf{A} \). We obtain:
\[ H_E = \frac{\varepsilon_0}{2} \int dV |E(r,t)|^2 \]
\[ = \frac{\varepsilon_0}{2} \int dV \mathbf{E}^*(r,t) \cdot \mathbf{E}(r,t) \]
\[ = \frac{\varepsilon_0}{2} \sum \sum \sum \sum \int dV (-i\omega_k) (i\omega_{k'}) \left[ A_{k\lambda}e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} - A^*_{k\lambda}e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \right] \]
\[ \times \left[ A^*_{k\lambda}e^{i\omega_{k'} t - i\mathbf{k'} \cdot \mathbf{r}} - A_{k\lambda}e^{-i\omega_{k'} t + i\mathbf{k'} \cdot \mathbf{r}} \right] \hat{e}_{k'\lambda'} \cdot \hat{e}_{k\lambda} \]  
(1.34)

**NOTE:** The only quantities that remain inside of the integral are the exponentials.

By recalling that
\[ \int dV e^{\pm i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} = V \delta_{k,k'}, \]  
(1.35a)
and
\[ \int dV e^{\pm i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}} = V \delta_{k,-k'}, \]  
(1.35b)
we obtain, substituting the above result into Eq. (1.20),

\[
H_E = -\sum_{kk'}\sum_{\lambda\lambda'} \frac{V\varepsilon_0\omega_k\omega_{k'}}{2} A_{k',\lambda'}^* A_{k,\lambda} e^{i(\omega_{k'}-\omega_k)t\delta_{k,k'}} \hat{e}_{k',\lambda'} \cdot \hat{e}_{k,\lambda} + \\
+ A_{k',\lambda'}^* A_{k,\lambda} e^{i(\omega_{k'}+\omega_k)t\delta_{k,\lambda}} \hat{e}_{k',\lambda} \cdot \hat{e}_{k,\lambda} + A_{k',\lambda'} A_{k,\lambda} e^{-i(\omega_{k'}+\omega_k)t\delta_{k,\lambda}} \hat{e}_{k',\lambda} \cdot \hat{e}_{k,\lambda} \\
+ A_{k',\lambda'} A_{k,\lambda} e^{-i(\omega_{k'}-\omega_k)t\delta_{k,k'}} \hat{e}_{k',\lambda'} \cdot \hat{e}_{k,\lambda}
\]

\[
= -\sum_k \sum_\lambda \varepsilon_0 V\omega_k^2 |A_{k,\lambda}|^2 - \sum_k \sum_\lambda \frac{V\varepsilon_0\omega_k^2}{2} [A_{k,\lambda}^* A_{-k,\lambda'} \hat{e}_{-k,\lambda'} \cdot \hat{e}_{k,\lambda} e^{i2\omega_k t} + \\
+ A_{k,\lambda}^* A_{-k,\lambda'} \hat{e}_{-k,\lambda'} \cdot \hat{e}_{k,\lambda} e^{-i2\omega_k t}] .
\] (1.36)

The second term in Eq. (1.36) compensates a similar term in the magnetic field Hamiltonian, which is given by:

\[
H_B = -\frac{1}{2\mu_0} \sum_{kk'} \sum_{\lambda\lambda'} \int dV \left[ A_{k',\lambda'} e^{-i\omega_{k'} t - ik' r} + A_{k',\lambda'}^* e^{i\omega_{k'} t - ik' r} \right] \\
\times \left[ A_{k,\lambda}^* e^{i\omega_k t - ik r} - A_{k,\lambda} e^{-i\omega_k t + ik r} \right] (\hat{e}_{k',\lambda'} \times (\hat{e}_{k,\lambda} ) ) (\hat{e}_{k,\lambda} )
\] (1.37)

Analogously to the electric field Hamiltonian, the integration gives factors \( \delta_{k,k'} \) and \( \delta_{k,-k'} \). With this we obtain:

\[
\delta_{k,k'} \rightarrow (\hat{e}_{k,\lambda'} \hat{e}_{k,\lambda} ) = \hat{e}_{k,\lambda} = k^2 \delta_{\lambda\lambda'} \quad (1.38a)
\]

\[
\delta_{k,-k'} \rightarrow (\hat{e}_{k,\lambda'} \hat{e}_{k,\lambda} ) = -k^2 \hat{e}_{-k,\lambda} \cdot \hat{e}_{k,\lambda'} \quad (1.38b)
\]

Combining \( H_E \) and \( H_B \) we obtain the full electromagnetic field Hamiltonian

\[
H = \sum_k \sum_{\lambda=1}^2 \varepsilon_0 V\omega_k^2 (A_{k,\lambda} A_{k,\lambda}^* + A_{k,\lambda}^* A_{k,\lambda}).
\] (1.39)

For the calculation we have used that

\[
\varepsilon_0 \omega_k^2 \hat{e}_{-k,\lambda'} \cdot \hat{e}_{k,\lambda} - \frac{k^2}{\mu_0} \hat{e}_{k,\lambda} \cdot \hat{e}_{-k,\lambda'} = \left( \varepsilon_0 \omega_k^2 - \frac{k^2}{\mu_0} \right) \hat{e}_{k,\lambda} \cdot \hat{e}_{-k,\lambda'} = \\
\left( \varepsilon_0 \omega_k^2 - \frac{k^2}{\mu_0} \right) \frac{\varepsilon_0 c^2 k^2 - k^2}{\mu_0} = \frac{1}{\varepsilon_0 c^2 k^2 - k^2} = 0,
\] (1.40)

where \( \omega_k = ck \).

**NOTE:** Although for the classic EM field \( AA^* = A^* A \) we still write this as \( AA^* + A^* A \) instead of \( 2|A|^2 \) in order to obtain a direct connection with the quantized form, which we will see later, where \( A \) and \( A^* \) will be promoted to operators.


2 QUANTIZATION OF THE FREE ELECTROMAGNETIC FIELD

We consider the EM field Hamiltonian as a collection of cavity modes, as given by Eq. (1.39), i.e.,

\[ H = \sum_{k} \sum_{\lambda=1}^{2} \varepsilon_0 V \omega_k^2 (A_{k\lambda} A_{k\lambda}^* + A_{k\lambda}^* A_{k\lambda}). \]

We now introduce the creation and annihilation operators and express the Hamiltonian in terms of these. If we compare the above expression of the field Hamiltonian of a harmonic oscillator (see Eq. (10.3)), we see that we can obtain the quantized version of the field Hamiltonian if we promote the mode coefficients $A$ and $A^*$ to operators in such a way that $A \approx a^\dagger$ and $A^* \approx a$, namely

\[ A_{k\lambda} \rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \hat{a}^\dagger_{k\lambda}, \quad (2.1a) \]
\[ A_{k\lambda}^* \rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \hat{a}_{k\lambda}. \quad (2.1b) \]

Substituting this into Eq. (1.39) brings us the following form for the quantized field Hamiltonian:

\[ \hat{H} = \sum_{k,\lambda} \frac{\hbar \omega_k}{2} \left( \hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda} + \frac{1}{2} \right). \quad (2.2) \]

The following commutation rules are understood

\[ [\hat{a}_{k\lambda}, \hat{a}_{k'\lambda'}^\dagger] = \delta_{k,k'} \delta_{\lambda,\lambda'}, \quad (2.3a) \]
\[ [\hat{a}_{k\lambda}, \hat{a}_{k'\lambda'}] = 0 = [\hat{a}_{k\lambda}^\dagger, \hat{a}_{k'\lambda'}^\dagger]. \quad (2.3b) \]

If we make use of the above commutation relation, Eq. (2.2) can be written in the following, more inspiring, form:

\[ \hat{H} = \sum_{k,\lambda} \frac{\hbar}{2} \omega_{k,\lambda} \left( \hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda} + \frac{1}{2} \right) \]
\[ = \sum_{k} \sum_{\lambda=1}^{2} \hbar \omega_{k,\lambda} \left( \hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda} + \frac{1}{2} \right). \quad (2.4) \]

This result can be interpreted in a very intuitive way: thanks to the above equation, the total energy of the electromagnetic field can be interpreted as given by the sum of the contribution of several uncoupled harmonic oscillators, each one characterized by its own frequency $\omega_{k,\lambda}$. Since the field is composed by several uncoupled oscillators, it is reasonable to assume that for each oscillator (i.e., each field mode), there should be a state $|n_{k\lambda}\rangle$ associated. This allows us to write the global state of the
electromagnetic field as a direct product of all such states, i.e.,

\[
|n_{k_1,1}, n_{k_1,2}, n_{k_2,1}, n_{k_2,2}, \ldots \rangle = |n_{k_1,1}\rangle |n_{k_1,2}\rangle |n_{k_2,1}\rangle \ldots \equiv |n_{k\lambda}\rangle \quad (2.5)
\]

These states are known in literature as Fock states, and constitute the energy eigenstates of the electromagnetic field. To understand that, let us analyze how the Hamilton operator \(\hat{H}\) acts on such states, namely

\[
\hat{H} |n_{k\lambda}\rangle = \sum_k \sum_{\lambda=1}^2 \hbar \omega_k \left( \hat{a}_{k\lambda}^{\dagger} \hat{a}_{k\lambda} + \frac{1}{2} \right) |n_{k\lambda}\rangle
\]

\[
= \sum_k \sum_{\lambda=1}^2 \hbar \omega_k \left[ \hat{a}_{k\lambda}^{\dagger} \sqrt{n_{k\lambda}} |n_{k\lambda} - 1\rangle + \frac{1}{2} |n_{k\lambda}\rangle \right]
\]

\[
= \sum_k \sum_{\lambda=1}^2 \hbar \omega_k \left( n_{k\lambda} + \frac{1}{2} \right) |n_{k\lambda}\rangle , \quad (2.6)
\]

where the following relations have been used:

\[
\hat{a} |n\rangle = \sqrt{n} |n - 1\rangle \quad (2.7a)
\]

\[
\hat{a}^{\dagger} |n\rangle = \sqrt{n} |n - 1\rangle \quad (2.7b)
\]

**NOTE:** Combining the above relations we can introduce the photon number operator \(\hat{N}_{k\lambda}\) as follows

\[
\hat{N}_{k\lambda} = \hat{a}_{k\lambda}^{\dagger} \hat{a}_{k\lambda} . \quad (2.8)
\]

The matrix elements of the Hamilton operator of the free field are then obtained by multiplying eq. \((2.6)\) from the left side with \(\langle n_{k\lambda}|\) we obtain:

\[
\langle n_{k\lambda}| \hat{H} |n_{k\lambda}\rangle = \sum_k \sum_{\lambda=1}^2 \hbar \omega_k \left( n_{k\lambda} + \frac{1}{2} \right)
\]

\[
\quad (2.9)
\]

This is the total energy of the field expressed as the sum of quanta distributed among all the possible field modes. We used \(\langle n_{k\lambda}| n_{k'\lambda'}\rangle = \delta_{kk'} \delta_{\lambda\lambda'}\).

**IMPORTANT NOTE:**

- \(n_{k\lambda}\) is the number of quanta (field excitations - photons) in the mode \((k, \lambda)\);
- Classical Electromagnetism: NO FIELD = NO ENERGY;
- Quantum Optics: For the vacuum state \(n_{k\lambda} = 0\) we obtain:

\[
\hat{H} |0\rangle = \sum_k \sum_{\lambda=1}^2 \hbar \omega_k \left( n_{k\lambda} + \frac{1}{2} \right) |0\rangle = \sum_k \sum_{\lambda=1}^2 \frac{\hbar \omega_k}{2} \neq 0 . \quad (2.10)
\]
So the energy of the vacuum state is $\sum_{k,\lambda} \hbar \omega_k/2$. It must be nonzero because of the Heisenberg principle (compare to particle in a box)

To conclude the quantization procedure, let us now write the vector potential, the electric and the magnetic field operators in terms of $\hat{a}$ and $\hat{a}^\dagger$. The expressions of the vector potential, electric and magnetic fields in terms of the vector potential coefficients $A$ and $A^*$ are given as follows:

$$A(r, t) = \sum_k \sum_{\lambda=1}^2 \hat{e}_{k\lambda} \left[ A_{k\lambda} e^{-i\omega_k t + ik \cdot r} + A_{k\lambda}^* e^{i\omega_k t - ik \cdot r} \right], \quad (2.11a)$$

$$E(r, t) = \sum_k \sum_{\lambda=1}^2 \hat{e}_{k\lambda} \cdot i \omega_k \left[ A_{k\lambda} e^{-i\omega_k t + ik \cdot r} - A_{k\lambda}^* e^{i\omega_k t - ik \cdot r} \right], \quad (2.11b)$$

$$B(r, t) = \sum_k \sum_{\lambda=1}^2 i (k \times \hat{e}_{k\lambda}) \left[ A_{k\lambda} e^{-i\omega_k t + ik \cdot r} - A_{k\lambda}^* e^{i\omega_k t - ik \cdot r} \right]. \quad (2.11c)$$

By recalling that $A_{k\lambda} \rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \hat{a}_{k\lambda}^\dagger$ and $A_{k\lambda}^* \rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \hat{a}_{k\lambda}$, we can therefore write the field operators in the following form:

$$A(r, t) = A^+(r, t) + A^-(r, t), \quad (2.12a)$$

$$E(r, t) = E^+(r, t) + E^-(r, t), \quad (2.12b)$$

$$B(r, t) = B^+(r, t) + B^-(r, t). \quad (2.12c)$$

Where $+$ indicates the positive frequency part and $-$ the negative one. Furthermore:

$$A^+(r, t) = \sum_k \sum_{\lambda=1}^2 \hat{e}_{k\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \hat{a}_{k\lambda} e^{-i\omega_k t + ik \cdot r}, \quad (2.13a)$$

$$A^-(r, t) = \sum_k \sum_{\lambda=1}^2 \hat{e}_{k\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \hat{a}_{k\lambda}^\dagger e^{i\omega_k t - ik \cdot r}, \quad (2.13b)$$

$$E^+(r, t) = i \sum_k \sum_{\lambda=1}^2 \hat{e}_{k\lambda} \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}} \hat{a}_{k\lambda} e^{-i\omega_k t + ik \cdot r}, \quad (2.14a)$$

$$E^-(r, t) = -i \sum_k \sum_{\lambda=1}^2 \hat{e}_{k\lambda} \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}} \hat{a}_{k\lambda}^\dagger e^{i\omega_k t - ik \cdot r}, \quad (2.14b)$$

$$B^+(r, t) = i \sum_k \sum_{\lambda=1}^2 \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} (k \times \hat{e}_{k\lambda}) \hat{a}_{k\lambda} e^{-i\omega_k t + ik \cdot r}, \quad (2.15a)$$

$$B^-(r, t) = -i \sum_k \sum_{\lambda=1}^2 \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} (k \times \hat{e}_{k\lambda}) \hat{a}_{k\lambda}^\dagger e^{i\omega_k t - ik \cdot r}. \quad (2.15b)$$
3 QUANTUM STATES OF THE ELECTROMAGNETIC FIELD

We now want to study the possible states of the electromagnetic field and their properties. Without loss of generality, we consider only single mode fields, i.e., the electromagnetic field is represented by a single harmonic oscillator corresponding to a single degree of freedom of the field (polarization, frequency, spatial mode, etc.). To this aim, let us fix the polarization of the field to be $\hat{x}$ and let us assume that the field is characterised by a single $k$-vector $k$ and that it propagates along the $z$-direction. Using these assumptions, the electric field operator defined in Eq. (2.11b) assumes then the following form:

$$\hat{E}(z,t) = i E_0 [\hat{a} e^{i(kz-\omega t)} - \hat{a}^\dagger e^{-i(kz-\omega t)}],$$

(3.1)

where $E_0 = \sqrt{\hbar \omega / 2 \varepsilon_0 V}$. Notice, moreover, that for monochromatic plane wave fields $|B| = |E|/c$ and the magnetic field can be therefore be neglected.

3.1 Number States

According to the results of the previous section, the Hamiltonian of a single mode field is given by

$$\hat{H} = \hbar \omega \left( \hat{n} + \frac{1}{2} \right),$$

(3.2)

where $\hat{n} = \hat{a}^\dagger \hat{a}$ is the number operator, which counts the number of excitations in a mode, and therefore its eigenvalues $n$ give the number of excitations (i.e., photons) in the mode. It makes then sense to introduce the Fock (number) states as follows:

**Def:**

Fock (number) states are eigenstates of the number operator, such that

$$\hat{n} |n\rangle = n |n\rangle.$$ 

(3.3)

3.1.1 Properties of Fock States

Fock states form a complete set of orthonormal vectors, i.e.,

- Orthogonality:
  $$\langle n | m \rangle = \delta_{nm},$$

  (3.4)

- Completeness:
  $$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1.$$ 

(3.5)

The operators $\hat{a}^\dagger$ and $\hat{a}$ create and annihilate, respectively, one photon in the
field i.e.,
\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (3.6a) \]
\[ \hat{a} |n\rangle = \sqrt{n} |n-1\rangle. \quad (3.6b) \]

In order to create a state from the vacuum we need to apply this relations recursively:
\[ |n\rangle = (\hat{a}^\dagger)^n \sqrt{n!} |0\rangle, \quad (3.7) \]
while the vacuum state is defined as
\[ \hat{a} |0\rangle = 0. \quad (3.8) \]

The expectation value of the Hamilton operator of the field over the Fock states is given by
\[ H = \langle n | \hat{H} | n \rangle = \hbar \omega \left( n \langle n | n \rangle + \frac{1}{2} \langle n | n \rangle \right) \]
\[ = \hbar \omega \left( n \langle n | n \rangle + \frac{1}{2} \langle n | n \rangle \right) \]
\[ = \hbar \omega \left( n + \frac{1}{2} \right) \langle n | n \rangle, \quad (3.9) \]
and therefore:
\[ H = \hbar \omega \left( n + \frac{1}{2} \right). \quad (3.10) \]

The expectation value of the electric field operator (3.1) is instead given by
\[ \langle n | \hat{E}(z,t) | n \rangle = i \mathcal{E}_0 \left[ \langle n | \hat{a} | n \rangle e^{i(kz-\omega t)} - \langle n | \hat{a}^\dagger | n \rangle e^{-i(kz-\omega t)} \right] \]
\[ = i \mathcal{E}_0 \left[ \sqrt{n} e^{i(kz-\omega t)} \langle n | n-1 \rangle - \sqrt{n+1} \langle n | n+1 \rangle e^{-i(kz-\omega t)} \right] \]
\[ = 0. \quad (3.11) \]

The electric field is not well defined for number states!!
However \( \langle \hat{E} \rangle^2 \) is nonzero since it is connected with the field energy
\[ \langle n | \hat{E}^2 | n \rangle = 2 \mathcal{E}_0^2 \left( n + \frac{1}{2} \right). \quad (3.12) \]

We can therefore calculate the field variance:
\[ \left\langle \Delta \hat{E}(z,t) \right\rangle^2 = \left\langle \hat{E}^2(z,t) \right\rangle - \left\langle \hat{E}(z,t) \right\rangle^2 = 2 \mathcal{E}_0^2 \left( n + \frac{1}{2} \right). \quad (3.13) \]

The above equation has an interesting consequence: letting \( n = 0 \), i.e. no photons
in the field, the variance 

\[ \langle \Delta \hat{E}(z,t) \rangle^2 \neq 0. \]

This means that the vacuum state \(|0\rangle\) shows field fluctuations (quantum noise). Moreover the number operator and the electric field operator do not commute, i.e.,

\[ \left[ \hat{n}, \hat{E}(z,0) \right] = i \mathcal{E}_0 \left\{ -\hat{a}e^{i(kz-\omega t)} + \hat{a}^\dagger e^{-i(kz-\omega t)} \right\}. \] (3.14)

**Summarizing:**
- Number states are quantum states of well definite photon number. As a consequence, the electric field is not well defined for these states;
- vacuum field fluctuations.

### 3.2 Coherent states

As we have seen number states do not have a well defined \(E\) and therefore are not fitted to describe a classical field. So how can we find then a classical expression for a quantum state of the e.m. field?

**Considerations:**

1. Large numbers of photons in the field \(|N\rangle\), \(N \gg 1\)?

   **FALSE:**

   we just saw, that no matter how big \(N\) is, a number state has an ill-defined \(E\) and therefore the classical limit cannot come from \(|N\rangle\)

2. Since \(\hat{E} \sim \hat{a} - \hat{a}^\dagger\) a possible trial solution could be \(|\psi\rangle \sim |n\rangle + |n + 1\rangle\)?

   **Good:** we generalize this concept and therefore obtain the definition:

**Def. Coherent States:** Coherent states are eigenstates of the annihilation operator and therefore obtain following eigenvalue equation:

\[ \hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad \alpha \in \mathbb{C} \] (3.15)

\[ (\langle \alpha | \hat{a}^\dagger = \langle \alpha | \alpha^*). \]

Let us write the coherent states in number state basis:

\[ |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \] (3.16)

from this two equations we can calculate the expansion coefficients:

\[ \hat{a} |\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a} |n\rangle = \sum_{n=1}^{\infty} \sqrt{n} c_n |n - 1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle. \] (3.17)
Now, in order to let both sums on the r.h.s. of eq. (3.17) start from the same number, i.e. \( n = 1 \), we need to change the expression \( \sum_{n=0}^{\infty} c_n \ket{n} \) to \( \sum_{n=1}^{\infty} c_{n-1} \ket{n-1} \). This leads to

\[
\sqrt{n} c_n = \alpha c_{n-1}. \tag{3.18}
\]

Applying successively the annihilation operator on equation (3.17) it follows that

\[
c_n = c_0 \frac{\alpha^n}{\sqrt{n!}}, \tag{3.19}
\]

where \( c_0 \) is defined by normalization, i.e., by imposing \( \langle \alpha | \alpha \rangle = 1 \). This leads to

\[
\langle \alpha | \alpha \rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{\langle n | \alpha \rangle \langle \alpha | m \rangle \delta_{nm}}{\sqrt{n!m!}} = |c_0|^2 e^{\alpha |\alpha|^2}. \tag{3.20}
\]

Therefore:

\[
|\alpha\rangle = \exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \ket{n} = \exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{n=0}^{\infty} \frac{\alpha^n (\hat{a}^\dagger)^n}{\sqrt{n!}} \ket{0}. \tag{3.21}
\]

### 3.2.1 Expectation Value of the electric Field over Coherent States

Let us now calculate the expectation value of the electric field in coherent state basis, using

\[
\langle \alpha | \hat{E}(z,t) | \alpha \rangle = i \mathcal{E}_0 \langle \alpha | \hat{a} e^{i(kz-\omega t)} - \hat{a}^\dagger e^{-i(kz-\omega t)} | \alpha \rangle \tag{3.22}
\]

\[
= i \mathcal{E}_0 \left[ \alpha e^{i(kz-\omega t)} - \alpha^* e^{-i(kz-\omega t)} \right] \tag{3.23}
\]

\[
= i \mathcal{E}_0 |\alpha| \left[ e^{i(kz-\omega t+\phi)} - e^{-i(kz-\omega t+\phi)} \right] \tag{3.24}
\]

\[
= -2 \mathcal{E}_0 |\alpha| \sin(kz - \omega t + \phi), \tag{3.25}
\]

where in passing from the second to the third line we have used the fact that \( \alpha = |\alpha| \exp(i\phi) \). As can be seen, moreover, the expectation value of the electric field operator over the coherent state \( |\alpha\rangle \) gives as result the classical electric field.

Now let us calculate the variance of the electric field operator, i.e.,

\[
\langle \alpha | \Delta \hat{E}^2(z,t) | \alpha \rangle = \langle \alpha | \hat{E}^2(z,t) | \alpha \rangle - \left( \langle \alpha | \hat{E}(z,t) | \alpha \rangle \right)^2 \tag{3.26}
\]

\[
= \mathcal{E}_0^2 - \frac{\hbar \omega}{2 \varepsilon_0 V}.
\]

A coherent state is a nearly classical state because it reproduces correctly the classical field although it still contains field fluctuations (that are the same of the vacuum field). Note, moreover, coherent states are also defined for the harmonic oscillator as
states of minimal uncertainty.

3.2.2 Action of the Number Operator on Coherent States

Let us now investigate how the number operator acts on coherent states. Therefore we will calculate the expectation value of the number operator in coherent state basis, as follows

\[ \langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2 \langle \alpha | \alpha \rangle = |\alpha|^2 \equiv \bar{n}. \]  

(3.27)

This means that coherent states do not possess a well defined number of photons, as we will see now.

In fact, compared to the number states where the variance \( \langle \Delta \hat{n} \rangle^2 = 0 \), the variance of the photon number doesn’t vanish for coherent states. In order to convince ourselves, that this is the case, let us calculate

\[ \langle \Delta \hat{n} \rangle^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \langle \alpha | \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle - \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle. \]  

(3.28)

Here we are going to use the commutation rule

\[ [\hat{a}, \hat{a}^\dagger] = \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1, \]  

(3.29)

which leads to

\[ \langle \alpha | \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger (1 + \hat{a}^\dagger \hat{a}) \hat{a} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle + \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \alpha \rangle = |\alpha|^2 + |\alpha|^4. \]  

(3.30)

Combining our results and plugging them back into eq. (3.28), we obtain the variance of the photon number operator for the case of coherent states as follows:

\[ \langle \Delta \hat{n} \rangle^2 = |\alpha|^2 + |\alpha|^4 - |\alpha|^4 = |\alpha|^2, \]  

(3.31)

hence

\[ \langle \hat{n} \rangle = \langle \Delta \hat{n} \rangle^2 = |\alpha|^2 \]  

(3.32)
### 3.2.3 Photon Probability

The probability of finding \( n \) photons in a coherent state \(|\alpha\rangle\) can be calculated as follows:

\[
P_n = |\langle n | \alpha \rangle|^2 = \left| \exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \langle n | m \rangle \right|^2
\]

\[
= \exp \left[ -|\alpha|^2 \right] \frac{(|\alpha|^2)^n}{n!} = \exp \left[ -\bar{n}^2 \right] \frac{\bar{n}^n}{n!}.
\]

This satisfies the Poissonian statistic (mean value = variance):

\[
P_n = \frac{\bar{n}^n}{n!} \exp[-\bar{n}^2]
\]

and moreover:

\[
\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{n}}
\]

### 3.2.4 Summary

**Number States:**

Number states are described by \(|n\rangle\). They are eigenstates of the number operator \(\hat{n} = \hat{a}^\dagger \hat{a}\) and fulfill the equation

\[
\hat{n} |n\rangle = n |n\rangle.
\]

The mean value of the electric field in number state basis is:

\[
\langle n | \hat{E} | n \rangle = 0
\]

but the variance is:

\[
\left\langle \Delta \hat{E} \right\rangle^2 \sim \left( n + \frac{1}{2} \right) \quad \text{(quantum noise)}
\]

**Coherent States:**

Coherent states are the most classical among the quantum states. They are described by \(|\alpha\rangle\). They are eigenstates of the creation operator \(\hat{a}\) and fulfill the equation

\[
\hat{a} |\alpha\rangle = \alpha |\alpha\rangle
\]

We can transform them to number state basis by:

\[
|\alpha\rangle = \exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\]
The mean value of the electric field in coherent state basis is:

\[ \langle \hat{E} \rangle = -2\mathcal{E}_0 |\alpha| \sin (kz - \omega t + \phi) \]

and the variance:

\[ \langle \Delta \hat{E} \rangle = \mathcal{E}_0 \equiv \sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} \]

### 3.3 Phase Space and Quadrature Operators for the Electromagnetic Field

Classical mechanics: \( H(q, p) \rightarrow \) each point in phase space represents a configuration of the system (see fig. 3.1)

\[ H(q, p) \rightarrow \text{each point in phase space represents a configuration of the system} \]

![Figure 3.1: Representation of the system configuration in phase space](image)

Figure 3.1: Representation of the system configuration in phase space

We can introduce this also in quantum optics. In order to do this, we first remember the harmonic oscillator and adapt the same rules to our operator:

\[ \hat{x} \sim \hat{a} + \hat{a}^\dagger \rightarrow \hat{X}_1 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger) \quad (3.36a) \]

\[ \hat{p} \sim \hat{a} - \hat{a}^\dagger \rightarrow \hat{X}_2 = \frac{1}{2i} (\hat{a} - \hat{a}^\dagger) \quad (3.36b) \]

Now we can build a 2D phase space associated to the single mode quantum field using \( \hat{X}_1 \) and \( \hat{X}_2 \), with

\[ [\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \quad (3.37) \]
3.3.1 Number states in phase space

Expectation value:

\[ \langle n | \hat{X}_1 | n \rangle = 0 = \langle n | \hat{X}_2 | n \rangle \]  \hspace{1cm} (3.38)

but variance:

\[ \langle n | \Delta \hat{X}_1^2 | n \rangle = \frac{1}{4} (2n + 1) = \langle n | \Delta \hat{X}_2^2 | n \rangle \]  \hspace{1cm} (3.39)

Proof:

\[
\langle \Delta \hat{X}_1^2 \rangle = \langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2 \\
= \langle n | \frac{1}{4} (\hat{a} + \hat{a}^\dagger) (\hat{a} + \hat{a}^\dagger) | n \rangle \\
= \frac{1}{4} \langle n | \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^\dagger | n \rangle \\
= \frac{1}{4} \left[ \langle n | \hat{a} \sqrt{n + 1} | n + 1 \rangle + \langle n | \hat{a}^\dagger \sqrt{n} | n - 1 \rangle \right] \\
= \frac{1}{4} [(n + 1) \langle n | n \rangle + n \langle n | n \rangle] = \frac{1}{4} (2n + 1) \]  \hspace{1cm} (3.40)

Since the variances are non-zero and \([\hat{X}_1, \hat{X}_2] \neq 0\) we can apply an indetermination relation

\[ \Delta \hat{X}_1^2 \Delta \hat{X}_2^2 \geq \frac{1}{16} \]  \hspace{1cm} (3.41)

\[ \rightarrow \Delta \hat{X}_1 \neq 0, \text{ even if } n = 0 \text{ (vacuum fluctuations). Moreover:} \]

\[ \langle \Delta \hat{X}_1^2 \rangle + \langle \Delta \hat{X}_2^2 \rangle = \frac{1}{2} (2n + 1) = n + \frac{1}{2} = \langle n | \hat{n} + \frac{1}{2} | n \rangle, \]  \hspace{1cm} (3.42)

the vacuum fluctuations are automatically taken into account.

\[ \text{Phase Space Representation(fig.3.2)} \]

- circle of radius \( \sqrt{n + 1/2} \) (radius is well defined)
- phase completely undefined (circle)
- vacuum state = circle with radius \( \sqrt{1/2} \)

Figure 3.2: Representation of number states in phase space

3.3.2 Coherent states in phase space

For coherent states we have a peculiar thing:

\[ \langle \alpha | \hat{X}_1 | \alpha \rangle = \frac{1}{2} (\alpha + \alpha^*) = \text{Re} \{\alpha\} \]  \hspace{1cm} (3.43a)
\[ \langle \alpha | \hat{X}^2 | \alpha \rangle = \frac{1}{2i} (\alpha - \alpha^*) = \text{Im}\{\alpha\} \]  

(3.43b)

This means that the complex \( \alpha \)-plane and the phase space for a coherent state are isomorph with each other. Therefore it is useful to use the complex polar representation for \( \alpha \), i.e.:

\[ \alpha = |\alpha| e^{i\theta}. \]  

(3.44)

The phase space representation of a coherent state is illustrated in Fig. 3.3

Figure 3.3: Representation of coherent states in phase space

Moreover:

\[ \langle \Delta \hat{X}_1^2 \rangle = \langle \alpha | \hat{X}_1^2 | \alpha \rangle - \left( \langle \alpha | \hat{X}_1 | \alpha \rangle \right)^2 = \frac{1}{4} = \langle \Delta \hat{X}_2^2 \rangle \]  

(3.45)

the variance of the quadrature operator is the same as the quantum vacuum. Especially:

\[ \Delta \hat{X}_1 \Delta \hat{X}_2 = \frac{1}{4} \]  

(3.46)

and therefore coherent states are states of minimum uncertainty. **Note:** Looking at the figure, one can imagine (also motivated by the fact, that coherent states have the same \( \Delta \hat{x}_k \) as the vacuum state) that coherent states can be viewed as suitably displaced vacuum states.

Remember:

\[ \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2 \]

there is no well defined number of photons for a coherent state. Furthermore there is also a phase uncertainty (because of Heisenberg uncertainty principle)
3.4 Displacement Operator

How is this possible?

Let’s assume that there is an operator $\hat{D}(\alpha)$ such that:

$$\hat{D}(\alpha) |0\rangle = |\alpha\rangle. \quad (3.47)$$

To obtain an expression for $\hat{D}(\alpha)$ let us note that

$$|\alpha\rangle = \exp \left[ -\frac{|\alpha|^2}{2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right]$$

With:

$$|n\rangle = \sum_{n=0}^{\infty} \frac{\hat{a}\dagger^n}{\sqrt{n!}} |0\rangle$$

are tempted to say that eq. (3.48) describes the displacement operator. In order to understand why this is the case let us consider the **Baker-Campbell-Hausdorff-Formula**:

$$e^{\hat{A}+\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]} e^{\hat{B}} e^{\hat{A}} \quad (3.50)$$
let us choose:
\[ \hat{A} = \alpha \hat{a}^\dagger \text{ and } \hat{B} = \alpha^* \hat{a}, \]
then we obtain:
\[ [\hat{A}, \hat{B}] = [\alpha \hat{a}^\dagger, \alpha^* \hat{a}] = |\alpha|^2 [\hat{a}^\dagger, \hat{a}] = |\alpha|^2 \]
However, we are still missing a factor \( e^{\alpha^* \hat{a}} \). But note that:
\[
e^{-\alpha^* \hat{a}} |n\rangle = \sum_{m=0}^{\infty} \frac{(-\alpha \hat{a})^m}{m!} |0\rangle = \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{m!} (\hat{a})^m |0\rangle.
\]
(3.51)
If we therefore define
\[ \hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \]
as the displacement operator, the factor \( e^{\alpha^* \hat{a}} \) has no effect but contributes to make the displacement operator \( \hat{D}(\alpha) \) more symmetric.

**Note:** The displacement operator is unitary:

- \( \hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) \)
- \( \hat{D}(\alpha) \hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha) \hat{D}(\alpha) = 1 \)

**Note:** You can prove, that without \( e^{\alpha^* \hat{a}} \) the displacement operator is not unitary!

**More on \( \hat{D}(\alpha) \):** It is useful and constructive to calculate the following quantity
\[
\langle \alpha | \hat{a} | \alpha \rangle = \langle 0 | \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) | 0 \rangle
\]
**What is then \( \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) \)?**
It is hard to calculate this by hand, but we will use the:

**Decomposition Theorem for Exponential Operators**
\[
e^{\hat{A} \hat{B} e^{-\hat{A}}} = \hat{B} + \left[ \hat{A}, \hat{B} \right] + \frac{1}{2!} \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] + \frac{1}{3!} \left[ \hat{A}, \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] \right] + \ldots \quad (3.53)
\]
For our purpose we chose: \( \hat{A} = -\alpha \hat{a}^\dagger + \alpha^* \hat{a} \) and \( \hat{B} = \hat{a} \). So we obtain:
\[
\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + [-\alpha \hat{a}^\dagger + \alpha^* \hat{a}, \hat{a}]
\]
\[= \hat{a} + \left\{ -\alpha \left[ \hat{a}^\dagger, \hat{a} \right] + \alpha^* \left[ \hat{a}, \hat{a} \right] \right\}
\]
\[= \hat{a} + \alpha \quad (3.54)
\]
Summarizing:

\[
\hat{D}^\dagger (\alpha) \hat{a} \hat{D} (\alpha) = \hat{a} + \alpha \\
\hat{D}^\dagger (\alpha) \hat{a}^\dagger \hat{D} (\alpha) = \hat{a}^\dagger + \alpha^* 
\]

(3.55)  
(3.56)

In this way we can evaluate each expectation value as expectation value done on the vacuum state. And so we obtain, that:

\[
\langle \alpha | f \left( \hat{a}, \hat{a}^\dagger \right) | \alpha \rangle = \langle 0 | \hat{D}^\dagger (\alpha) f \left( \hat{a}, \hat{a}^\dagger \right) \hat{D} (\alpha) | 0 \rangle \\
= \langle 0 | f \left( \hat{a} + \alpha, \hat{a}^\dagger + \alpha^* \right) | 0 \rangle 
\]

(3.57)

**Coherent States:** Classical field \((\alpha)\) plus quantum fluctuations.

**Squeezed States:** States of the electromagnetic field, for which it is possible for a quadrature to beat the Heisenberg limit. Of course the price to pay is that the uncertainty on the other quadrature gets bigger.

### 3.5 Squeezed States

So far we have considered number states (well defined \(n\) but ill-defined \(\hat{E}\) and \(\hat{B}\)) and coherent states (almost classical, minimal uncertainty) and their phase space representations (see Fig. 3.3). They fulfil the uncertainty relation:

\[
\Delta \hat{X}_1 \Delta \hat{X}_2 \geq \frac{1}{4}. 
\]

(3.58)

Is this the limit for manipulating states or is there another possibility to play around beating the fundamental Heisenberg limit?

![Figure 3.5: Comparison of a squeezed state with the vacuum state in phase space](image)

As before \(\Delta \hat{X}_1 \Delta \hat{X}_2 = \text{const}\) but in the certain case the uncertainty on \(x_2\) is much less than in the cases before.

How can we define squeezed states and how are they generated?
We seek for a state $|\xi\rangle$ with $\Delta \hat{X}_1^2 \Delta \hat{X}_2^2 \geq \frac{1}{16}$, but:

\[
\begin{align*}
\Delta \hat{X}_1^2 &= \frac{1}{4} \gamma, \\
\Delta \hat{X}_2^2 &= \frac{1}{4\gamma},
\end{align*}
\] (3.59) (3.60)

where $\gamma \leq 1$ so that it is a reduction of the variance (uncertainty) for $x_1$ but at the same time it is an amplification for the variance of $x_2$ (so that the product is still constant).

We employ:

1. Since we learned, that every (coherent) state can be displaced from the vacuum, let us consider the vacuum itself and squeeze it (Squeezed Vacuum, actually this is, what we generate in the lab).

2. Let us assume to be able to define an operator $\hat{S}(\xi)$ similar to the displacement operator $\hat{D}(\alpha)$, namely

\[
\hat{S}(\xi) = e^{\frac{1}{2}[\xi \hat{a}^2 - \xi (\hat{a}^\dagger)^2]}.
\] (3.61)

Eq. (3.61) is the squeezing operator, where $\xi \in \mathbb{C}$ and $\xi = re^{i\vartheta}$.

![Figure 3.6: Schematic illustration of squeezed vacuum](image)

Note: For the squeezed state in fig 3.6 we have $\Delta \hat{X}_1 < \frac{1}{2}$, $\Delta \hat{X}_2 > \frac{1}{2}$ but $\Delta \hat{X}_1 \Delta \hat{X}_2 = \frac{1}{4}$ (For more information on dynamics in phase space see Ref. [4]).

Note: $\hat{S}(\xi)$ contains $\hat{a}^2$ and $(\hat{a}^\dagger)^2$, i.e. the squeezing operator is a nonlinear operator. If we want to generate squeezed states, therefore, we must do it through a...
nonlinear process (typically SPDC). Furthermore we require \( \hat{S}(\xi) \) to be unitary:
\[
\hat{S}^\dagger(\xi) = \hat{S}(-\xi),
\]
(3.62)
\[
\hat{S}^\dagger(\xi) \hat{S}(\xi) = \hat{S}(\xi) \hat{S}^\dagger(\xi) = 1.
\]
(3.63)

Therefore we can generate a squeezed vacuum state by applying the squeezing operator on the vacuum state:
\[
|\xi\rangle = \hat{S}(\xi)|0\rangle.
\]
(3.64)

The variance of \( \hat{X}_1 \) is obtained by
\[
\langle \Delta \hat{X}_1^2 \rangle = \langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2.
\]
(3.65)

For this we calculate the expectation value of \( \hat{X}_1 \):
\[
\langle \xi | \hat{X}_1 | \xi \rangle = \frac{1}{2} \langle \xi | \hat{a} + \hat{a}^\dagger | \xi \rangle = \frac{1}{2} \langle 0 | \hat{S}^\dagger(\xi) (\hat{a} + \hat{a}^\dagger) \hat{S}(\xi) | 0 \rangle.
\]
(3.66)

As we did for the displacement operator for the coherent states, we look here for:
\[
\hat{S}^\dagger(\xi) \hat{a} \hat{S}(\xi) = \hat{a} \cosh r - \hat{a}^\dagger e^{i\vartheta} \sinh r,
\]
(3.67)
\[
\hat{S}^\dagger(\xi) \hat{a}^\dagger \hat{S}(\xi) = \hat{a}^\dagger \cosh (r) - \hat{a} e^{i\vartheta} \sinh r.
\]
(3.68)

Therefore we obtain:
\[
\langle \xi | \hat{X}_1 | \xi \rangle = \cosh r \langle 0 | \hat{a} | 0 \rangle - e^{i\vartheta} \sinh r \langle 0 | \hat{a}^\dagger | 0 \rangle +
\]
\[
\cosh r \langle 0 | \hat{a}^\dagger | 0 \rangle - e^{i\vartheta} \sinh r \langle 0 | \hat{a} | 0 \rangle = 0.
\]
(3.69)

For \( \vartheta = 0 \) we obtain at the end:
\[
\langle \xi | \Delta \hat{X}_1^2 | \xi \rangle = \frac{1}{4} e^{-2r} \rightarrow \langle \xi | \Delta \hat{X}_2^2 | \xi \rangle = \frac{1}{4} e^{2r},
\]
(3.70)

r is then called the "squeezing parameter".

**Final Notes:**
\[
\hat{a} | 0 \rangle = 0,
\]
but then also:
\[
\hat{S}(\xi) \hat{a} | 0 \rangle = 0.
\]

Since \( \hat{S}(\xi) \) is unitary and therefore \( \hat{S}^\dagger(\xi) \hat{S}(\xi) = 1 \) and therefore:
\[
\hat{S}(\xi) \hat{a} | 0 \rangle = \hat{S}(\xi) \hat{a} \hat{S}^\dagger(\xi) \hat{S}(\xi) | 0 \rangle = \hat{S}(\xi) \hat{a} \hat{S}^\dagger(\xi) | \xi \rangle = 0.
\]
(3.71)

This allows us to define the squeezed states as eigenstates of the operator \( \hat{S}(\xi) \hat{a} \hat{S}^\dagger(\xi) \).
with the eigenvalue zero. **To Summarize:**

**Number States:** \( \hat{n} |n \rangle = n |n \rangle \)

**Coherent States:** \( \hat{a} |\alpha \rangle = \alpha |\alpha \rangle \)

**Squeezed States:** \( \hat{S}(\xi) \hat{a} \hat{S}^\dagger(\xi) |\xi \rangle = 0 \)

### 4 Interaction of Matter with Quantized Light -I

After studying the free electromagnetic field and introducing its quantum states, the next step is to consider the interaction of the E.M. field with matter. Up to now we considered the E.M. field in free space, therefore the first step is to rewrite the electromagnetic potentials in presence of matter. Afterwards we will introduce atoms to our calculations and then analyse their interaction with the quantized field. As a result of our considerations we will be able to introduce the interaction Hamiltonian of the E.M. field with matter at the end of this section.

#### 4.1 Electromagnetic Potentials in Presence of Matter

Since we are not considering the case of vacuum anymore we need to take into account charge and current densities. Therefore the E.M. potentials in presence of matter are now given as follows

\[
\nabla (\nabla \cdot A) - \nabla^2 A + \frac{1}{c^2} \frac{\partial \nabla \phi}{\partial t} + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \mu_0 J
\]

(4.1a)

\[
- \nabla^2 \phi = \frac{\sigma}{\varepsilon_0}
\]

(4.1b)

\[
\nabla \cdot A = 0 \quad \text{(Coulomb Gauge)}
\]

(4.1c)

In general eqs. (4.1a) and (4.1c) are difficult to solve, since we cannot say anymore that \( \phi = 0 \). In presence of matter the scalar potential \( \phi(r) \) is given as

\[
\phi(r) = \frac{1}{\varepsilon_0} \int \frac{\sigma(r')}{|r-r'|} dr'.
\]

(4.2)

Moreover we know that since \( \nabla A = 0 \) the vector potential \( A \) needs to be transversal, and so the correspondent \( E \) field must also be transversal. In order to separate the transversal and longitudinal part of a vector field we use the Helmholtz theorem. The Helmholtz theorem states that it is possible to split any vector field in three dimensions into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field. Applying the H.T. on eq. (4.1a) we obtain for the current \( J \) that

\[
J = J_L + J_T,
\]

(4.3)
with
\[ \nabla J_T = 0 \text{ (solenoidal)}, \]  
\[ \nabla \times J_L = 0 \text{ (irrotational)}. \]  
(4.4a, 4.4b)

The Coulomb gauge fixes the vector potential to be transverse. So the \( \mu_0 J_L \) term does not contribute to \( A \). Therefore we obtain after separating the longitudinal and transverse parts:
\[ -\nabla^2 A + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \mu_0 J_T, \]  
(4.5a)
\[ \frac{1}{c^2} \frac{\partial \nabla \phi}{\partial t} = \mu_0 J_L, \]  
(4.5b)
\[ -\nabla^2 \phi = \frac{\sigma}{\epsilon_0}. \]  
(4.5c)

From the last two equations we obtain the continuity equation
\[ \nabla \cdot J_L = -\frac{\partial \sigma}{\partial t}. \]  
(4.6)

Splitting \( J \) in its transversal and longitudinal parts allows us to split the fields in the same way, namely
\[ E = E_T + E_L, \]  
(4.7a)
\[ B = B_T + B_L, \]  
(4.7b)

with
\[ E_T = -\frac{\partial \sigma}{\partial t}, \quad B_T = \nabla \times A, \]  
(4.8)
\[ E_L = -\nabla \phi, \quad B_L = 0. \]  
(4.9)

While the longitudinal field is basically the electrostatic field generated by the charge distribution \( \sigma \), the transverse field is the one that propagates. \( B_L = 0 \) because of \( \nabla \cdot B = 0 \).

### 4.2 Introduction of Atoms

Let us now specify our analysis to the case of an atom (with \( Ne^- \), see Fig. [4.1]) as a field source and try to solve equations (4.5).
4.2.1 Charge Density

All the electrons and protons are treated like point charges, this is why we will write the charge density $\sigma(r)$ as

$$\sigma(r) = -e \sum_{\alpha} \delta(r - r_{\alpha}) + eZ\delta(r).$$  \hspace{1cm} (4.10)

This charge density generates a longitudinal field $E_L$, which is essentially the Coulomb potential of the atom).

4.2.2 Current Density

Each electron orbits the nucleus, this is why we write the total current density $J(r)$ as a sum of the currents produced by each electron

$$J(r) = -e \sum_{\alpha} \dot{r}_{\alpha}\delta(r - r_{\alpha}),$$  \hspace{1cm} (4.11)

where the dot stands for the time derivative.
4.3 Atom-Field Interaction

It is convenient to treat the interaction of the atom with the electromagnetic field in terms of the polarization \( \mathbf{P}(\mathbf{r}) \) and magnetization \( \mathbf{M}(\mathbf{r}) \), since they are associated with the macroscopic atomic charges and currents. Using the standard results of electromagnetism (see, e.g., [3]), the polarization and magnetization are linked to the charge density and current (respectively) via the following relations:

\[
\sigma(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}), \quad (4.12)
\]

\[
\mathbf{J}(\mathbf{r}) = \dot{\mathbf{P}}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r}). \quad (4.13)
\]

The polarization \( P \) can be thought to be the sum of all the contributions coming from the dipoles composing the medium. In our case, such elementar dipoles are represented by each single electron and the atom’s nucleus. The polarization of such an elementar dipole is then given by

\[
\mathbf{P} = q_\alpha \mathbf{r}_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) \quad (4.14)
\]

In order to simplify our calculations we will use a trick in the choice of expressing the polarization, namely we imagine to cover the distance between the nucleus (situated at \( r = 0 \)) and the electron at \( \mathbf{r}_\alpha \) with a series of non-point dipoles oriented along \( \mathbf{r}_\alpha \), in such a way that \( -q_\alpha \) of one dipole is compensated by \( +q_\alpha \) of the preceding one (see Fig. 4.2).

Figure 4.2: Split polarization vector between nucleus and electron

Therefore, since for a single dipole we have that

\[
\mathbf{P}_n(\mathbf{r}) = (q_\alpha \mathbf{r}_\alpha/n) \delta \left( \mathbf{r} - \frac{p + \frac{1}{2} \mathbf{r}_\alpha}{n} \right), \quad (4.15)
\]

summing all the contributions as in Fig. 4.2 and taking the limit \( n \to \infty \) gives
\[ P(r) = -e \sum_a r_\alpha \int_0^1 d\xi \delta(r - \xi r_\alpha). \quad (4.16) \]

Reasoning in an analogous way for \( M \) we obtain:

\[ M(r) = -e \sum_a r_\alpha \times \dot{r}_\alpha \int_0^1 d\xi \delta(r - \xi r_\alpha). \quad (4.17) \]

Recalling that potential energy of a permanent electric and a permanent magnetic dipole are given by \(-d \cdot E\) and \(-\mu \cdot B\), respectively, we will define the electric \( V_E \) and magnetic \( V_M \) contribution to the atom’s potential energy as

\[ V_E = -\int d^3r P(r) \cdot E_T(r) = e \sum_a \int_0^1 d\xi r_\alpha \cdot E_T(\xi r_\alpha), \quad (4.18) \]

and

\[ V_M = -\int d^3r M(r) \cdot B(r) = -e \sum_a \int_0^1 d\xi (r_\alpha \times \dot{r}_\alpha) \cdot B(\xi r_\alpha). \quad (4.19) \]

Usually the typical scale of atoms (a few Ångström) is much smaller than the wavelength of light (400 to 700 nm), therefore let us now expand the fields around \( \xi r_\alpha = 0 \) in order to evaluate the various interaction contributions:

**4.3.1 Electric Part**

\[ V_E = e \sum_a \int_0^1 d\xi \left\{ 1 + \xi (r_\alpha \cdot \nabla) + \frac{1}{2!} \xi^2 (r_\alpha \cdot \nabla)^2 + \ldots \right\} r_\alpha \cdot E_T(0). \quad (4.20) \]

Here we can perform the \( \xi \)-Integration and obtain:

\[ V_E = e \sum_a \left\{ \underbrace{1}_{\text{dipole}} + \underbrace{\frac{1}{2!} (r_\alpha \cdot \nabla)}_{\text{quadrupole}} + \underbrace{\frac{1}{3!} (r_\alpha \cdot \nabla)^2}_{\text{hexapole}} + \ldots \right\} r_\alpha \cdot E_T(0). \quad (4.21) \]

Equation (4.21) is the multipole expansion of the electric field potential. In particular we notice that

**Dipole:** \(-eD = -\sum_\alpha e r_\alpha \Rightarrow eD \cdot E_T(0)\)

**Quadrupole:** \(Q = -\frac{1}{2} \sum_\alpha e r_\alpha r_\alpha \Rightarrow -\nabla \cdot Q \cdot E_T(0)\)
4.3.2 Magnetic Part

Now we will do the same expansion for the magnetic field.

\[
V_M = \frac{e}{m} \sum_\alpha \left\{ \frac{1}{2!} (r_\alpha \cdot \nabla) I_\alpha \cdot B(0) + \frac{2}{3!} (r_\alpha \cdot \nabla)^2 I_\alpha \cdot B(0) \right\}
\]

where \( I_\alpha = m r_\alpha \times \frac{d}{dt} \) is the angular momentum of the \( \alpha \)th electron. Eq. (4.22) is then a multipole expansion of the magnetic field potential. The explicit expression of the magnetic dipole, e.g., is then given by

\[
- e D_M = - \frac{1}{2} \sum_\alpha \frac{e}{m} I_\alpha \Rightarrow e D_M \cdot B(0)
\]

4.3.3 Orders of Magnitude

In order to get an idea of which contributions of the multipole expansion we have to take in account in our calculations, let us consider the following argument. To do that, let us take, approximately, \( r_\alpha \approx a_0 = 10^{-10} \text{m} \) (Bohr radius), \( I_\alpha \approx \hbar = 10^{-34} \) (quantum of angular momentum) and \( \nabla E_T(0) \approx \omega c E_T(0) \), \( \omega \approx 3 \cdot 10^{15} \text{Hz} \) (visible light). If we substitute these numbers in the definitions of electric dipole and quadrupole, as well as magnetic dipole we obtain:

\[
\begin{align*}
\cdot d_e &= e D \cdot E_T(0) \approx \frac{4\pi\varepsilon_0 h^2}{mc} E_T(0) \\
\cdot q_e &= -\nabla \cdot Q \cdot E_T(0) \approx \frac{3e h}{16mc} E_T(0) \\
\cdot d_m &= e D_M \cdot B(0) \approx \frac{eh}{2mc} B(0) \approx \frac{eh}{2mc} E_T(0) \quad \text{(plane wave: } B \sim \frac{E}{c})
\end{align*}
\]

If we now compare the different coefficients with each other:

1. \( \frac{q_e}{d_m} = \frac{3e h E_T(0)}{16mc} \cdot \frac{2mc}{e h E_T(0)} = \frac{3}{8} \approx 1 \),

i.e., the electric quadrupole and magnetic dipole terms are approximately of the same order of magnitude

2. \( \frac{q_e}{d_e} = \frac{3e h E_T(0)}{16mc} \cdot \frac{me}{4\pi\varepsilon_0 h^2 E_T(0)} \approx \frac{e^2}{4\pi\varepsilon_0 hc} \sim \frac{1}{137} \equiv \alpha \),

i.e., the electric dipole is roughly 2 orders of magnitude bigger than the quadrupole (and \( \alpha^n \) orders of magnitude bigger than the n-th pole). It is therefore also \( \alpha \)-times bigger than the magnetic dipole!

**Conclusion:** It makes sense to consider interactions of an atom with an electromagnetic field in the dipole approximation, since this is the term that gives the
biggest contribution.

**Note:** Some cases forbid the transition to the electric dipole (selection rules). In all those cases we need to consider contributions from higher order of the electric multipole expansion, or, if the atom is magnetically active, then take into account the magnetic multipole expansion.

### 4.4 The Interaction Hamiltonian

Now we are going to discuss, how to construct the dipole-Hamiltonian out of our theory. We are interested in quantized fields, so we need to take the approach of quantizing \( E \) through quantizing \( A \). We therefore need to write \( D \cdot E \) starting from \( A \).

**Hamiltonian of the Atom-Light System**

Since we still want to have a Hamiltonian representation, we need to take into account the canonical momentum of the system. Furthermore, we still want to be able to use the canonical quantization scheme to quantize the field.

In our first attempt let us assume that the Hamiltonian we are looking for has this form:

\[
\mathcal{H}^{(1)} = \sum_{\alpha} \frac{p_{\alpha}^2}{2m} + \frac{1}{2} \int d^3r \sigma(r)\phi(r) + \frac{1}{2} \int d^3r \left[ \frac{\varepsilon_0}{\mu_0} |E(r)|^2 + \frac{1}{\mu_0} |B(r)|^2 \right]
\]  

(4.24)

Considering this Hamiltonian first thing we notice is that there is no interaction part. We could add an interaction term by hand but which kind of? Moreover, it can be demonstrated, that \( p_\alpha \) is not a canonical momentum anymore when the field is there. Hence it is better to use the minimal coupling principle where \( p_\alpha \rightarrow \left[ \hat{p}_\alpha + e\hat{A}(r) \right] \).

This definition takes automatically into account the atom-field interaction through the vector potential \( A(r) \). As a consequence of this consideration our new Hamiltonian should have following form:

\[
\mathcal{H}' = \frac{1}{2m} \sum_{\alpha} \left[ \hat{p}_\alpha + e\hat{A}(r) \right]^2 + \frac{1}{2} \int d^3r \sigma(r)\phi(r) + \frac{1}{2} \int d^3r \left[ \varepsilon_0 \hat{E}_T^2(\mathbf{r}) + \frac{1}{\mu_0} \hat{B}^2(\mathbf{r}) \right]
\]  

(4.25)

Where the first term is the Hamiltonian of a free particle in the presence of an electromagnetic field. The second term is the Coulomb potential of the atom (completely classical, no operators). Finally the third term is the Hamiltonian of the free electromagnetic field.
Let us develop the first term of eq. (4.25) to extract the interaction Hamiltonian:

\[ \frac{1}{2m} \sum_{\alpha} \left[ \hat{p}_\alpha + e\hat{A}(r) \right]^2 = \sum_{\alpha} \frac{\hat{p}^2_\alpha}{2m} + \frac{e}{m} \sum_{\alpha} \hat{A}(r_\alpha) \hat{p}_\alpha + \left( \frac{e^2}{2m} \right) \sum_{\alpha} \hat{A}^2(r_\alpha). \] (4.26)

Since the first term on the right hand side of this equation describes the kinetic energy of the isolated atom, we obtain the interaction Hamiltonian as:

\[ \hat{H}'_I = \frac{e}{m} \sum_{\alpha} \hat{A}(r_\alpha) \hat{p}_\alpha + \left( \frac{e^2}{2m} \right) \sum_{\alpha} \hat{A}^2(r_\alpha). \] (4.27)

**Note:**

- The interaction Hamiltonian is written as a function of \( A \) and is therefore not gauge invariant
- We want gauge invariance and will write it therefore as a function of \( E \cdot D \)

For this we need to apply the so called Power-Zienau-Wooley-transformation

\[ \hat{U} = \exp \left[ \frac{i}{\hbar} \int d^3r \hat{p}(r) \hat{A}(r) \right]. \] (4.28)

The application of \( \hat{U} \) to the system Hamiltonian is such that

\[ \hat{H} = \hat{U}^\dagger \hat{H}' \hat{U}. \] (4.29)

In doing that, however, the state \( |\psi'\rangle \) of the system must also be transformed according to

\[ |\psi\rangle = \hat{U}^\dagger |\psi'\rangle. \] (4.30)

After some calculation we obtain the full Hamiltonian describing the atom-field interaction:

\[ \hat{H} = \sum_{\alpha} \frac{\hat{p}^2_\alpha}{2m} + \frac{1}{2} \int d^3r \sigma(r) \phi(r) + \frac{1}{2} \int d^3r \left[ \hat{E}^2_T(r) + \frac{1}{\mu_0} \hat{B}^2(r) \right] + e \sum_{\alpha} r_\alpha \cdot \hat{E}_T(0), \] (4.31)

where now the interaction Hamiltonian is in the right dipole form:

\[ \hat{H}_{ED} = e \sum_{\alpha} r_\alpha \cdot \hat{E}_T(0) = e \mathbf{D} \cdot \mathbf{E}_T(0). \]
4.5 Second Quantization

4.5.1 Second Quantization of the Atomic Hamiltonian

We have seen (see eq. (4.31)) that the atomic Hamiltonian is given by:

\[ \hat{H}_A = \sum_\alpha \frac{p_\alpha^2}{2m} + \frac{1}{2} \int d^3r \, \sigma(r) \phi(r). \]

This form, however, is not useful to us, since we would prefer a second quantized approach, i.e., we would like to express the atomic Hamiltonian in terms of "creation" and "annihilation" operators, so that we can unify the formalism with the one used for describing the electromagnetic field. To do so, we consider the atomic energy eigenstates \( |i\rangle \), with eigenvalues \( \hbar \omega_i \), so that:

\[ \hat{H}_A |i\rangle = \hbar \omega_i |i\rangle. \quad (4.32) \]

Here we will use the completeness of \( |i\rangle \), i.e., \( 1 = \sum_i |i\rangle \langle i| \) in order to write

\[
\hat{H}_A = \left( \sum_i |i\rangle \langle i| \right) \hat{H}_A \left( \sum_j |j\rangle \langle j| \right)
\]

\[
= \sum_{ij} |i\rangle \langle i| \hat{H}_A |j\rangle \langle j| \]
\]

\[
= \sum_{ij} |i\rangle \hat{H}_A \langle i| \langle j| \]
\]

\[
= \sum_{ij} \hbar \omega_i \delta_{ij} |i\rangle \langle i| 
\]

\[
= \hbar \omega_i |i\rangle \langle i|. \quad (4.33) 
\]

The second quantized form of the atomic Hamiltonian is then given by:

\[ \hat{H}_A = \sum_i \hbar \omega_i |i\rangle \langle i| \quad (4.34) \]

In this form, \( \hat{H}_A \) is written as a sum of projections onto the energy eigenstates \( |i\rangle \), where each of them has the eigenvalue \( \hbar \omega_i \). In order to catch the essential physics behind the atom-field interaction in a simple and intuitive way, we will consider the case of a two-level system, i.e., we assume the atomic system only to possess a ground state, \( |1\rangle \), and an excited state, \( |2\rangle \). The transition frequency is defined as \( \omega_0 = \omega_2 - \omega_1 \). A pictorial representation of such a system is given in Fig. 4.3.
The Hamiltonian of a two level system is then given by:

\[ \hat{H}_A = \hbar \omega_1 \langle 1 | + \hbar \omega_2 \langle 2 |. \]  

(4.35)

For a two level system it is useful to introduce the following operators:

\[ \hat{\sigma}^- = \langle 1 | 2 \rangle \]  

(4.36a)

\[ \hat{\sigma}^+ = (\hat{\sigma}^-)^\dagger = \langle 2 | 1 \rangle. \]  

(4.36b)

To understand their meaning, let us consider how they act on the atomic states, namely:

\[ \hat{\sigma}^- | 1 \rangle = \langle 2 | 1 \rangle \langle 1 | = 0. \]  

(4.37a)

\[ \hat{\sigma}^- | 2 \rangle = \langle 1 | 2 \rangle \langle 2 | = | 1 \rangle. \]  

(4.37b)

\[ \hat{\sigma}^+ | 1 \rangle = \langle 2 | 1 \rangle \langle 1 | = | 2 \rangle. \]  

(4.37c)

\[ \hat{\sigma}^+ | 2 \rangle = \langle 2 | 1 \rangle \langle 2 | = 0. \]  

(4.37d)

Then, \( \hat{\sigma}^+ \) brings the atom from \( | 1 \rangle \) to \( | 2 \rangle \), while \( \hat{\sigma}^- \) brings the atom from \( | 2 \rangle \) to \( | 1 \rangle \). Moreover

\[ \hat{\sigma}^- \hat{\sigma}^+ = \langle 2 | 2 \rangle \langle 1 | = | 1 \rangle \langle 1 |. \]  

(4.38a)

\[ \hat{\sigma}^+ \hat{\sigma}^- = \langle 2 | 1 \rangle \langle 1 | = | 2 \rangle \langle 2 |. \]  

(4.38b)

i.e., this two different combinations of the operators project directly into the eigenstates of the system. It is also useful to introduce the population inversion operator, \( \hat{\sigma}_3 \), as:

\[ \hat{\sigma}_3 = | 2 \rangle \langle 2 | - | 1 \rangle \langle 1 |. \]  

(4.39)

Then, the set \( \{ \hat{\sigma}^+, \hat{\sigma}^-, \hat{\sigma}_3 \} \) obeys the usual Pauli algebra:

\[ [\hat{\sigma}^+, \hat{\sigma}^-] = \hat{\sigma}_3. \]  

(4.40a)
\[ [\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm. \] (4.40b)

The atomic Hamiltonian can be then rewritten in terms of \( \hat{\sigma}_\pm \) as follows:

\[ \hat{H}_A = \hbar \omega_1 |1\rangle \langle 1| + \hbar \omega_2 |2\rangle \langle 2| = \hbar \omega_1 \hat{\sigma}_- \hat{\sigma}_+ + \hbar \omega_2 \hat{\sigma}_+ \hat{\sigma}_-. \] (4.41)

**Convention:** Since the energy is usually defined up to a constant, we have the freedom to decide where to put the zero of the energy. We have two choices:

1. \( E = 0 \) for the ground state (Fig. 4.3). Therefore we obtain:

\[ \hat{H}_A = \hbar \omega_0 \hat{\sigma}_+ \hat{\sigma}_-, \]

with \( \omega_0 = \omega_2 \).

2. \( E = 0 \) in the middle of the two levels (see fig. 4.4). Here we obtain \( \omega_1 = -\frac{\omega_0}{2} \) and \( \omega_0 = \frac{\omega_0}{2} \).

![Figure 4.4: Sketch of two level system with E = 0 between the two states](image)

If not specified explicitly, in the rest of this notes we will use the second convention. The Hamiltonian is therefore given by:

\[ \hat{H}_A = -\frac{\hbar \omega_0}{2} |1\rangle \langle 1| + \frac{\hbar \omega_0}{2} |2\rangle \langle 2| = \frac{\hbar \omega_0}{2} \hat{\sigma}_3. \]

### 4.5.2 Dipole Operator

It is now easy to second quantize the dipole operator. If we assume that the considered two level system has a transition allowed to the electric dipole then the representation of this operator in second quantization must be such that it realized the transition \( |1\rangle \rightarrow |2\rangle \) (and vice versa for the complex conjugate). If we call \( D_{12} = D_{21} \in \mathbb{R} \) the dipole moment of the atom (assuming it real), we obtain the dipole operator:

\[ \hat{D} = D_{12} (\hat{\sigma}_+ + \hat{\sigma}_-). \] (4.42)
4.5.3 Jaynes-Cummings Model

We have now all the elements to build a consistent model of the interaction of a two level atom with a quantized field. If we recall that the (single-mode) electric field operator is given by

\[ \hat{E}(z,0) = iE_0 (\hat{a}e^{ikz} - \hat{a}^\dagger e^{-ikz}) \hat{x}, \] (4.43)

\[(t = 0 \rightarrow \text{Schrödinger picture, see below})\] we can then rewrite \(\hat{H}_{\text{ED}}\), in the following way:

\[ \hat{H}_{\text{ED}} = e\hat{D} \cdot \hat{E} = ieE_0 (D_{12} \cdot \mathbf{x}) (\hat{a}e^{ikz} - \hat{a}^\dagger e^{-ikz}) (\hat{\sigma}_+ + \hat{\sigma}_-). \]

If we introduce

\[ \hbar g = eE_0 (D_{12} \cdot \mathbf{x}), \] (4.44)

where \(g\) describes the strength of the interaction, we obtain, assuming that the atom is locates at position \(z = 0\),

\[ \hat{H}_{\text{ED}} = \hbar g (\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- + \hat{a}^\dagger \hat{\sigma}_+ + \hat{a} \hat{\sigma}_-). \] (4.45)

What is the meaning of the 4 terms appearing in (4.45)? To understand that, let us consider the diagrammatic representation of the atom-field interaction given by Fig. 4.5, where

![Figure 4.5: Interaction of electric field with two level system](image)

the photon is illustrated as a solid line (the arrow shows in which direction the photon is propagating), the dashed line symbolizes the time evolution of the two level system and the atom is represented by a black dot. The initial state of the atom is on the right side of the diagram (at \(t = -\infty\)), while the final state is on the left side (at \(t = \infty\)). We assume for simplicity that the interaction occurs at \(t = 0\).

Now we will make sketches, analogous to Fig. 4.5, for all four terms of Eq. (4.45), in order to figure out which terms are contributing to the interaction:
Then, only $\hat{a} \hat{\sigma}_+$ and $\hat{a}^\dagger \hat{\sigma}_+$ give contributions to our calculations and the interaction Hamiltonian can be written as follows:

$$
\hat{H}_{ED} = \hbar g (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger).
$$

(4.46)
The full Hamiltonian in the Jaynes-Cummings model is then given by:

$$\hat{H} = \frac{1}{2} \hbar \omega_0 \hat{\sigma}_3 + \hbar \omega \hat{a}^\dagger \hat{a} + \hbar g \left( \hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right),$$  \hspace{1cm} (4.47)

together with the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1, \hspace{1cm} (4.48a)$$

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3, \hspace{1cm} (4.48b)$$

$$[\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2 \hat{\sigma}_\pm, \hspace{1cm} (4.48c)$$

and the other commutators are zero.

### 4.5.4 States Of The Joint System

The form of the Hamiltonian \((4.47)\) suggests that the considered system can be thought as the direct product of two disjoint Hilbert spaces, a 2D Hilbert space \(H_A\) belonging to the atom and spanned by \(\{|1\rangle, |2\rangle\}\), and an infinite-dimensional Hilbert space describing the e.m. field \(H_E\) and spanned, e.g. by the Fock states \(\{|n\rangle\}\).

Since these two Hilbert spaces are disjointed, one can define the state of the joint system atom + field simply as the direct product between these two spaces, i.e., \(H_J = H_A \otimes H_E\). The joint Hilbert space \(H_J\) is then spanned by \(\{|n, i\rangle = |n\rangle |i\rangle\}\).

### 4.6 Schrödinger, Heisenberg and Interaction Picture

So far we considered all the operators to be time-independent. The question is how to introduce the time dependence in our model. We have three possibilities:

#### 4.6.1 Schrödinger Picture

The Schrödinger picture is typically used in quantum mechanics. Here operators are time-independent

$$\frac{d\hat{O}_S}{dt} = 0, \hspace{1cm} (4.49)$$

whereas states evolve in time according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi_S (t)}{\partial t} = \hat{H} \Psi_S (t). \hspace{1cm} (4.50)$$

The formal solution is given by

$$\Psi_S (t) = e^{-\frac{i\hat{H}}{\hbar}} \Psi_S (0). \hspace{1cm} (4.51)$$

and \(d\hat{H}/dt = 0\).
4.6.2 Heisenberg Picture

Operators vary in time, while the states are time independent (typical in quantum field theory and quantum optics), namely

\[ \frac{d\Psi_H}{dt} = 0, \]

and the operators evolve according to the Heisenberg equation:

\[ i\hbar \frac{d\hat{O}_H}{dt} = [\hat{O}_H, \hat{H}] . \]

(4.52)

The total Hamiltonian, however, is time independent in the Heisenberg picture! The connection to the Schrödinger picture is given as:

\[ \Psi_H = e^{i\hat{H}t/\hbar} \Psi_S (t) = \Psi_S (0) , \]

(4.53)

and

\[ \hat{O}_H (t) = e^{i\hat{H}t/\hbar} \hat{O}_Se^{-i\hat{H}t/\hbar} . \]

(4.54)

Proof:

\[ \langle \Psi (t) | \hat{O}_S | \Psi (t) \rangle = \langle \Psi (0) | e^{i\hat{H}t/\hbar} \hat{O}_Se^{-i\hat{H}t/\hbar} | \Psi (0) \rangle = \langle \Psi (0) | \hat{O}_H (t) | \Psi (0) \rangle \]

4.6.3 Interaction Picture

Intermediate picture where the time evolution is shared between the state and the operator. In the interaction picture the wave function is defined as:

\[ \Psi_I (t) = e^{i\hat{H}_0 t/\hbar} \Psi_S (t) = e^{i\hat{H}_0 t/\hbar} e^{-i\hat{H}_0 t/\hbar} \Psi_S (0) , \]

(4.55)

where \( \hat{H}_0 \) is the free Hamiltonian of the total system. For the case of a two-level system interacting with the e.m. field, e.g., \( \hat{H}_0 = \hat{H}_A + \hat{H}_F \). The equation of motion is then given by:

\[ i\hbar \frac{d\Psi_I (t)}{dt} = i\hbar \left\{ \frac{i\hat{H}_0 t}{\hbar} e^{i\hat{H}_0 t/\hbar} \Psi_S (t) + e^{i\hat{H}_0 t/\hbar} \frac{d\Psi_S (t)}{dt} \right\} \]

\[ = -\hat{H}_0 \Psi_I (t) + e^{i\hat{H}_0 t/\hbar} \left\{ \hat{H}\Psi_S (t) \right\} = -\hat{H}_0 \Psi_I (t) + \hat{H}\Psi_S (t) \]

\[ = \left( -\hat{H}_0 + \hat{H}_0 + \hat{H}_ED \right) \Psi_S (t) . \]

And therefore:

\[ i\hbar \frac{d\Psi_I (t)}{dt} = \hat{H}_ED \Psi_S (t) \]

(4.56)
In the interaction picture the time evolution associated to operators is realized by making the Schrödinger operator evolving with the free Hamiltonian:

\[ \hat{O}_I = e^{i\hat{H}_0 t} \hat{O} e^{-i\hat{H}_0 t}, \]  

(4.57)

and accordingly:

\[ i\hbar \frac{d\hat{O}_I(t)}{dt} = \left[ \hat{O}_I(t), \hat{H}_0 \right]. \]  

(4.58)

The interaction picture is useful because all the dynamics are held by the interaction Hamiltonian. The free Hamiltonian only accounts for a global phase factor. We can therefore employ the Heisenberg representation to solve the atom-field dynamics. In this case, the time-evolution of the atomic and field operators is given by

\[ \hat{\sigma}_\pm \rightarrow \begin{cases} 
\hat{\sigma}_-(t) = \hat{\sigma}_- e^{-i\omega_0 t}, \\
\hat{\sigma}_+(t) = \hat{\sigma}_+ e^{i\omega_0 t}, 
\end{cases} \]  

(4.59)

and

\[ \hat{a}, \hat{a}^\dagger \rightarrow \begin{cases} 
\hat{a}(t) = \hat{a} e^{-i\omega t}, \\
\hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t}. 
\end{cases} \]  

(4.60)

The time dependent interaction Hamiltonian thus becomes:

\[ \hat{H}_{ED}(t) = \hbar g \left( \hat{\sigma}_+ \hat{a} e^{i(\omega_0 - \omega)t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i(\omega_0 - \omega)t} \right). \]  

(4.61)

Note: We assume \( \omega \approx \omega_0 \) (nearly resonant field). Having eliminated \( \hat{\sigma}_- \hat{a} \) and \( \hat{\sigma}_+ \hat{a}^\dagger \) corresponding to eliminate the terms that oscillate with \( \omega + \omega_0 \approx 2\omega_0 \) (rotating wave approximation).

5 Interaction of Matter with Quantized Light - II

5.1 Photon Emission and Absorption Rates

Let us now evaluate the matrix elements for absorption and emission processes driven by \( \hat{H}_{ED} \). Here, \( |i\rangle \) and \( |f\rangle \) describe the initial and final state of the joint system, respectively.

5.1.1 Absorption

First of all, let us have a look at the absorption process. Let us assume, that at the beginning of this process, the atom is in its ground state and the e.m. field has \( n \) photons. The absorption one photon of the e.m. field leads to the excitation of the atom. Therefore let us write the initial and final joint state for the absorption
Interaction of Matter with Quantized Light - II

process

|\ i\rangle = |n, 1\rangle = |n\rangle |1\rangle , \quad (5.1a)
|\ f\rangle = |n - 1, 2\rangle = |n - 1\rangle |2\rangle . \quad (5.1b)

Then

\[ \langle f| \hat{\text{H}}_{\text{ED}}(t)|i\rangle = \hbar g \left\{ e^{i(\omega_0 - \omega)t} \langle f| \hat{\sigma}_+ \hat{a} |i\rangle + e^{-i(\omega_0 - \omega)t} \langle f| \hat{\sigma}_- \hat{a}^\dagger |i\rangle \right\} \]
\[ = \hbar g \left\{ e^{i(\omega_0 - \omega)t} \langle n - 1| \hat{a} |n\rangle \langle 2| \hat{\sigma}_+ |1\rangle + e^{-i(\omega_0 - \omega)t} \langle n - 1| \hat{a}^\dagger |n\rangle \langle 2| \hat{\sigma}_- |1\rangle \right\} \]. \quad (5.2)

Using

\[ \langle 2| \hat{\sigma}_+ |1\rangle = \langle 2| 2\rangle \langle 1| 1\rangle = 1, \quad (5.3a)\]
\[ \langle n - 1| \hat{a} |n\rangle = \sqrt{n} \langle n - 1| n - 1\rangle = \sqrt{n}, \quad (5.3b)\]

we finally obtain

\[ \langle f| \hat{\text{H}}_{\text{ED}}(t)|i\rangle_{\text{ABS}} = \hbar g \sqrt{n} e^{i(\omega_0 - \omega)t}. \quad (5.4)\]

Comment: If there are no photons in the field \((n = 0)\), then \(\langle f| \hat{\text{H}}_{\text{ED}}(t)|i\rangle_{\text{ABS}} = 0\) (the atom cannot absorb photons). This is in accordance with the result from the semi-classical theory of light-matter interaction.

5.1.2 Emission

For the emission process we will consider an atom in its excited state and an e.m. field with \(n\) photons, i.e., the initial and final joint states of the emission process are given as

|\ i\rangle = |n, 2\rangle = |n\rangle |2\rangle , \quad (5.5a)
|\ f\rangle = |n + 1, 1\rangle = |n + 1\rangle |1\rangle . \quad (5.5b)

Then

\[ \langle f| \hat{\text{H}}_{\text{ED}}(t)|i\rangle = \hbar g \left\{ e^{i(\omega_0 - \omega)t} \langle f| \hat{\sigma}_+ \hat{a} |i\rangle + e^{-i(\omega_0 - \omega)t} \langle f| \hat{\sigma}_- \hat{a}^\dagger |i\rangle \right\} \]
\[ = \hbar g \left\{ e^{i(\omega_0 - \omega)t} \langle n - 1| \hat{a} |n\rangle \langle 2| \hat{\sigma}_+ |1\rangle + e^{-i(\omega_0 - \omega)t} \langle n - 1| \hat{a}^\dagger |n\rangle \langle 2| \hat{\sigma}_- |1\rangle \right\} \]. \quad (5.6)

Using

\[ \langle 1| \hat{\sigma}_- |2\rangle = \langle 1| 1\rangle \langle 2| 2\rangle = 1, \quad (5.7a)\]
and

\[ \langle n + 1| \hat{a} |n\rangle = \sqrt{n + 1} \langle n + 1| n + 1\rangle = \sqrt{n + 1}, \quad (5.7b)\]
we obtain
\[ \langle f | \hat{H}_{ED} (t) | i \rangle_{EMISS} = \hbar g \sqrt{n + 1} e^{-i(\omega_0 - \omega)t}. \] (5.8)

Comment: Since the matrix element depends on \( \sqrt{n + 1} \) it is different from zero also in the case in which there are no photons in the field! This means that an atom in the excited state can interact with the vacuum state of the field, resulting in decaying by emitting a photon (spontaneous emission).

5.1.3 Rate Equation

In order to derive the rate equations we will have a look at a two level system in a single-mode e.m. field. The time evolution of the population changes due to continuous absorption and emission of photons. This change is given by:

\[ \frac{dN_1}{dt} = -W_{ABS} N_1 + W_{EMISS} N_2, \] (5.9a)

\[ \frac{dN_1}{dt} = -W_{ABS} N_1 + W_{EMISS} N_2, \] (5.9b)

where \( N_1 \) is the population of level 1, \( N_2 \) describes the population of level 2, \( W_{ABS} \) stands for the absorption probability and \( W_{EMISS} \) gives the emission probability. In thermal equilibrium we have:

\[ \frac{dN_1}{dt} = 0 = \frac{dN_2}{dt} \] (5.10)

and therefore:

\[ W_{ABS} N_1 = W_{EMISS} N_2 \] (5.11)

And so:

\[ \frac{N_1}{N_2} = \frac{W_{EMISS}}{W_{ABS}} = \frac{\langle f | \hat{H}_{ED} (t) | i \rangle_{EMISS}}{\langle f | \hat{H}_{ED} (t) | i \rangle_{ABS}} = \frac{n + 1}{n}. \] (5.12)

At thermal equilibrium Boltzmann statistics imposes that

\[ \frac{N_1}{N_2} = \exp \left[ \frac{\hbar \omega_0}{kT} \right]. \] (5.13)

Comparing this two equations leads to

\[ \frac{n + 1}{n} = \exp \left[ \frac{\hbar \omega_0}{kT} \right], \] (5.14)

which fulfills the Bose-Einstein-Statistics

\[ n = \frac{1}{\exp \left[ \frac{\hbar \omega_0}{kT} \right] - 1}. \] (5.15)
5.2 Quantum Rabi Oscillations

5.2.1 The Semiclassical Rabi Model in a Nutshell

In order to have a comparison case, let us first briefly revise the results and the main dynamics of the semiclassical Rabi model. Assuming a monochromatic classical electric field \( E = E_0 \cos \omega t \), the semiclassical dipole Hamiltonian can be written as \( \hat{H}_{sc}(t) = \hat{V}_0 \cos \omega t \), where \( \hat{V}_0 = -\hat{d} \cdot E_0 \). The time dependent state vector can be written as a superposition (with time dependent coefficients) of the ground and excited states as follows:

\[
|\psi(t)\rangle = C_1(t)e^{-i\omega_1 t}|1\rangle + C_2(t)e^{-i\omega_2 t}|2\rangle.
\]

(5.16)

Using the Schrödinger equation we can obtain a set of two differential equations describing the evolution of the time dependent coefficients (that represent the amplitude probability of being in the correspondent level). The result is the following:

\[
\dot{C}_1 = -\frac{iV}{\hbar}e^{(\omega - \omega_0)t}C_2,
\]

(5.17a)

\[
\dot{C}_2 = \frac{iV}{\hbar}e^{-i(\omega - \omega_0)t}C_1,
\]

(5.17b)

where \( V = \langle 2 | -\hat{d} \cdot E_0 | 1 \rangle \in \mathbb{R} \) and \( \omega_0 = \omega_2 - \omega_1 \) is the transition frequency. In order to obtain the previous equations, moreover, we assumed that the electromagnetic field is nearly resonant with the atomic transition. With this approximation the terms oscillating at a frequency \( (\omega + \omega_0) \) have been neglected (Rotating Wave Approximation [2]).

Solutions of this system of differential equations can be sought for example by taking the derivative of the second equation and substituting the first one into it, thus obtaining the equation of motion for the probability amplitude of the excited state

\[
\ddot{C}_2 + i(\omega - \omega_0)\dot{C}_2 + \frac{V^2}{4\hbar^2}C_2 = 0.
\]

(5.18)

Solution of this second order equation with the initial condition that the ground state is the only populated state \( (C_1(0) = 1 \) and \( C_2(0) = 0 \) brings to the following-expressions for the probability amplitudes \( C_1(t) \) and \( C_2(t) \):

\[
C_1(t) = e^{i\Delta t/2} \left\{ \cos \left( \frac{\Omega_R t}{2} \right) - \frac{i\Delta}{\Omega_R} \sin \left( \frac{\Omega_R t}{2} \right) \right\},
\]

(5.19a)

\[
C_2(t) = \frac{iV}{\hbar\Omega_R}e^{i\Delta t/2} \sin \left( \frac{\Omega_R t}{2} \right),
\]

(5.19b)

where \( \Delta = \omega - \omega_0 \) is the detuning between the electric field frequency and the atomic transition, and the quantity \( \Omega_R \) is known as Rabi frequency and its expression is the
The Rabi frequency represents the frequency at which the oscillation of population between the ground and the excited state of the atom takes place, triggered by the interaction with the electromagnetic field. The probability that the atom is in the excited state is in fact given by

\[ P_2(t) = |C_2(t)|^2 = \frac{V^2}{\hbar^2 \Omega_R^2} \sin^2 \left( \frac{\Omega_R t}{2} \right), \]  

which for the case of perfect detuning (\( \Delta = 0 \)) assumes the easier form

\[ P_2(t) = \sin^2 \left( \frac{Vt}{2\hbar} \right). \]  

The time evolution of the population probability of the excited state is reported in Fig. 5.1 for various values of \( \Delta \). In the case of perfect resonance, a complete transfer of population between the ground and the excited state takes place at the time \( t = \pi \hbar / V \). An electromagnetic field characterized by the frequency \( \omega \) that realized this condition is said to be a \( \pi \)-pulse.

It is also useful to introduce the population inversion probability

\[ W(t) = P_2(t) - P_1(t) = \sin^2 \left( \frac{Vt}{2\hbar} \right) - \cos^2 \left( \frac{Vt}{2\hbar} \right) = -\cos \left( \frac{Vt}{\hbar} \right). \]  

Figure 5.1: Plot of \( P_2(t) \) versus \( t \) for various values of the detuning \( \Delta \). Figure adapted from Ref. [2].
At the time $t = \pi \hbar / 2V$, one can note that the population inversion is $W(\pi \hbar / 2V) = 0$, and the population is coherently shared between the ground and the excited state, i.e., the state of the system is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} (|1\rangle + i|2\rangle).$$

### 5.2.2 The quantum Rabi Model

We then want to investigate if the process of Rabi oscillations is influenced by the quantization of the field or not. Before starting solving the problem of calculating the dynamics of the Jaynes-Cummings model, let us make some preliminary observations. First of all we assume that no electrons (or atomic species in general) are lost by the system, and that therefore the population of the atomic system must remain constant over time. If we define the electron number operator as

$$\hat{P}_E = \hat{\sigma}_- \hat{\sigma}_+ + \hat{\sigma}_+ \hat{\sigma}_- = |1\rangle\langle 1| + |2\rangle\langle 2| = 1,$$

then it is immediate to see that

$$[\hat{P}_E, \hat{H}] = 0.$$ (5.26)

Since we are considering a closed system, not only the energy is conserved, but also the total number of excitations is conserved. This means that whenever a photon gets annihilated the atom (in order to conserve energy) must absorb it and make the transition to the excited state, and every time a photon is created, the atom must decay to the ground state. We can formalize this by defining the following excitation number operator

$$\hat{N}_E = \hat{a}^\dagger \hat{a} + \hat{\sigma}_- \hat{\sigma}_+,$$

so that

$$[\hat{N}_E, \hat{H}] = 0.$$ (5.28)

This allows us to rewrite the total Jaynes-Cummings Hamiltonian as the sum of two parts, one depending on $\hat{P}_E$ and $\hat{N}_E$ and the other one containing the dipolar interaction. Without loss of generality, we can consider first the simple case of perfect resonance, thus putting $\Delta = \omega - \omega_0 = 0$. As we did for the semiclassical case, the state of the system can be written as a superposition of the initial state $|i\rangle \equiv |n,2\rangle$ and the final state $|f\rangle \equiv |n+1,1\rangle$ (i.e. we are considering the case in which the atom is excited and decays to the ground state by emitting a photon) as follows:

$$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle,$$ (5.29)
with the initial conditions imposing that $C_i(0) = 1$ and $C_f(0) = 0$. If we now use
the above expression for the state and the interaction Hamiltonian $H_\text{int}$ to solve the
the Schrödinger equation for the considered system, we obtain the following set of
coupled differential equations for the coefficients $C_i(t)$ and $C_f(t)$:

$$
\dot{C}_i = -ig\sqrt{n+1}C_f, \quad (5.30a)
$$

$$
\dot{C}_f = -ig\sqrt{n+1}C_i. \quad (5.30b)
$$

Taking the derivative of the first equation and eliminating $C_f(t)$ by using the second
one we obtain the following equation of motion for the amplitude probability $C_i(t)$:

$$
\ddot{C}_i + g^2(n+1)C_i = 0, \quad (5.31)
$$

whose solution compatible with the initial condition is

$$
C_i(t) = \cos \left[ \frac{\Omega(n)t}{2} \right], \quad (5.32a)
$$

$$
C_f(t) = -i \sin \left[ \frac{\Omega(n)t}{2} \right], \quad (5.32b)
$$

where $\Omega(n) = 2g\sqrt{n+1}$ is the quantum Rabi frequency, whose expression should be
compared with the semiclassical result (at perfect resonance) $\Omega_R = V/\hbar$. In order
to draw some conclusion, we can define also in this case the probabilities that the
system is (at a certain time $t$) in the initial or final state as

$$
P_i(t) = |C_i(t)|^2 = \cos^2 \left[ \frac{\Omega(n)t}{2} \right], \quad (5.33a)
$$

$$
P_f(t) = |C_f(t)|^2 = \sin^2 \left[ \frac{\Omega(n)t}{2} \right], \quad (5.33b)
$$

and the population inversion probability as

$$
W(t) = \langle \psi(t) | \hat{\sigma}_3 | \psi(t) \rangle = P_i(t) - P_f(t) = \cos \Omega(n)t, \quad (5.34)
$$

that, apart from a minus sign due to the different choice of initial condition, coincides
with the result of Eq. (5.23), where the semiclassical Rabi frequency $\Omega_R$ has to be
substituted by its quantum counterpart $\Omega(n)$.

In the semiclassical case, the Rabi frequency depends on the amplitude of the
electric field through $V = \langle 2 | - \hat{d} \cdot \mathbf{E}_0 | 1 \rangle$, and if the amplitude of the field is zero
(meaning $\text{vec} \mathbf{E}_0 = 0$), there is no field, the Rabi frequency is consequently zero
and there is no population oscillation. In the fully quantized model, however, the
Rabi frequency depends on the number of photons in the field through the quantity
$\sqrt{n+1}$. As can be noted, $\Omega(0) \neq 0$. This has a straightforward physical meaning.
If the electromagnetic field is in its vacuum state, the atom has then a nonzero
probability of spontaneously emitting a photon and subsequently re-absorbing it and then re-emitting it etc, originating the so-called vacuum Rabi fluctuations. This process, of course, is a manifestation of the quantum nature of the electromagnetic field and has no classical counterpart.

However, apart from this difference, the dynamics of an atom interacting with an electromagnetic field with a definite number of photons (i.e., in a Fock state) is very much similar to the semiclassical case, as it gives rise to a regular and periodic dynamics, namely the Rabi oscillations. This could be a bit confusing, since we said in the previous chapters that number states are among the most nonclassical states of the electromagnetic field (they have no definite electric field!). However, the induced dynamics are practically the semiclassical one. One would have expected this result by using a coherent state, that is the most classical state of the field. However, in this case, the dynamics is much richer, and very different from the semiclassical case.

5.2.3 Interaction of an Atom with a Coherent State

To analyze the interaction of a two level system with an electromagnetic field described by a coherent state, we must first of all generalize the solution given above to the case of an atom interacting with a field in a superposition of Fock states. Once we have this result, in fact, writing down the solution for the coherent state case will be very easy.

To do that we assume the atomic state at the time $t = 0$ to be in a superposition of ground and excited state

$$|\psi(0)\rangle_{\text{atom}} = C_1|1\rangle + C_2|2\rangle,$$

and the electromagnetic field to be in a superposition of number states

$$|\psi(0)\rangle_{\text{field}} = \sum_{n=0}^{\infty} C_n|n\rangle.$$

As we said before, the initial state of the joint system atom+field can be written as the direct product of the corresponding Hilbert spaces, resulting in

$$|\psi(0)\rangle = |\psi(0)\rangle_{\text{atom}} \otimes |\psi(0)\rangle_{\text{field}} = \left\{ \sum_{n=0}^{\infty} C_n|n\rangle \right\} \{C_1|1\rangle + C_2|2\rangle\}.$$

At the generic time $t$, the state of the electromagnetic field can only be changed of one photon, either created (emission) or annihilated (absorption) and therefore

$$|\psi(t)\rangle_{\text{field}} = \sum_{n=0}^{\infty} \{C_n|n\rangle + C_{n\pm1}|n \pm 1\}.$$
For the atom, instead, the coefficients $C_1$ and $C_2$ are time dependent and therefore

$$|\psi(t)\rangle_{atom} = C_1(t)|1\rangle + C_2(t)|2\rangle. \quad (5.39)$$

The joint state of the system atom+field at the generic time $t$ is therefore given by

$$|\psi(t)\rangle = |\psi(t)\rangle_{atom} \otimes |\psi(t)\rangle_{field}. \quad (5.40)$$

This state must be a solution of the Schrödinger equation. Therefore:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = i\hbar \sum_n \left( C_n|n\rangle + C_{n+1}|n+1\rangle + C_{n-1}|n-1\rangle \right) \left[ \hat{C}_1(t)|1\rangle + \hat{C}_2(t)|2\rangle \right], \quad (5.41)$$

and

$$\hat{H}_{ED}|\psi(t)\rangle = h\sum_n \left( \hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right) \left( C_n|n\rangle + C_{n+1}|n+1\rangle + C_{n-1}|n-1\rangle \right) \left[ \hat{C}_1(t)|1\rangle + \hat{C}_2(t)|2\rangle \right] \times \left[ C_1(t)|1\rangle + C_2(t)|2\rangle \right]$$

$$= h\sum_n \left\{ \sqrt{n}C_n C_1(t)|n-1,2\rangle + \sqrt{n+1}C_n C_2(t)|n+1,1\rangle + \sqrt{n+1}C_{n+1} C_1(t)|n,2\rangle + \sqrt{n}C_{n-1} C_2(t)|n,1\rangle \right\}, \quad (5.42)$$

where in the last line we neglected the terms containing $|n \pm 2\rangle$ since they represent states of the electromagnetic field that cannot be reached (since at maximum the number of photons in the field can change by one unity by assumption). If we want to obtain the equation of motion for $C_1(t)$ and $C_2(t)$ we have to project the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}_{ED}|\psi(t)\rangle \quad (5.43)$$

onto the four different states $|n,1\rangle$ (corresponding to the atom in the ground state and the field with $n$ photons), $|n,2\rangle$ (atom in the excited state and $n$ photons in the field), $|n+1,1\rangle$ (the atom decayed to the ground state by emitting a photon) and $|n-1,2\rangle$ (the atom absorbed a photon to go to the excited state). We then obtain the following four equations of motion:

$$C_n \ddot{C}_1(t) + g^2 n C_n C_1(t) = 0, \quad (5.44a)$$

$$C_{n+1} \ddot{C}_1(t) + g^2 (n+1) C_{n+1} C_1(t) = 0, \quad (5.44b)$$

$$C_n \ddot{C}_2(t) + g^2 (n+1) C_n C_2(t) = 0, \quad (5.44c)$$

$$C_{n-1} \ddot{C}_1(t) + g^2 n C_{n-1} C_1(t) = 0. \quad (5.44d)$$

These equations, together with the initial condition (5.37) (and after having properly shifted the indices so that it is possible to write the solution as a single summation
in $n$) bring the following result:

$$
|\psi(t)\rangle = \sum_{n=0}^{\infty} \left\{ C_2 C_n \cos \left( gt\sqrt{n+1} \right) - i C_1 C_{n+1} \sin \left( gt\sqrt{n+1} \right) \right\} |2\rangle 
+ \left[ -i C_2 C_{n-1} \sin \left( gt\sqrt{n} \right) + C_1 C_n \cos \left( gt\sqrt{n} \right) \right] |1\rangle |n\rangle.
$$

(5.45)

In general, this state is an entangled state, meaning that it is not possible anymore to write this state as a simple product between a state of the field and a state of the atomic system, as it is done in Eq. (5.37). Therefore, in the general case of interaction of the field with an atomic system, it does not make sense anymore to talk about the field and the atom, but rather we must speak of the atom-field system, as the general wave function is non-separable anymore (entangled).

If we assume that the atom is initially in the excited state (i.e., we set $C_1(0) = 0$ and $C_2(0) = 1$), the previous solution can be still written as a product between an evolved field state and the atomic states as follows:

$$
|\psi(t)\rangle = |\psi_1(t)\rangle |1\rangle + |\psi_2(t)\rangle |2\rangle,
$$

(5.46)

where

$$
|\psi_1(t)\rangle = -i \sum_{n=0}^{\infty} C_n \sin \left( gt\sqrt{n+1} \right) |n+1\rangle,
$$

(5.47a)

$$
|\psi_2(t)\rangle = \sum_{n=0}^{\infty} C_n \cos \left( gt\sqrt{n+1} \right) |n\rangle.
$$

(5.47b)

In this case we can calculate analytically the atomic inversion probability as

$$
W(t) = \langle \psi(t) | \hat{\sigma}_3 | \psi(t) \rangle 
= \langle \psi_2(t) | \psi_2(t) \rangle - \langle \psi_1(t) | \psi_1(t) \rangle
= \sum_{n=0}^{\infty} |C_n|^2 \cos \left( 2gt\sqrt{n+1} \right).
$$

(5.48)

We now have a very general expression for the atomic inversion as a function of the expansion coefficients $C_n$. If we now want to study the dynamics of the system under the action of a coherent state, we just have t substitute in the previous expression the expression for the expansion coefficients of a coherent state in terms of number states, namely

$$
C_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}.
$$

(5.49)

If we do that, eq. (5.48) becomes (we call $\bar{n} = |\alpha|^2$ as the mean number of photons...
Figure 5.2: (a) Atomic inversion with the field initially in a coherent state with $\bar{n} = 5$. (b) Same as (a) but showing the evolution for a longer time, beyond the first revival time. Both graphs are plotted against the normalized time $T = gt$. The figures are adapted from Ref. [2].

In the coherent state)

$$W(t) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \cos \left(2gt\sqrt{n + 1}\right).$$  \hspace{1cm} (5.50)

In Fig. 5.2 a plot of the atomic inversion is given. As it appears clear by simply
looking at the graph, the dynamics of the system in this case is profoundly different from the semiclassical case. Let us try to understand what are the characteristics of this dynamics. First of all, from Fig. 5.2(a) we note that the Rabi oscillation appears to damp out after a certain time. If we however wait long enough [Fig. 5.2(b)], the oscillations start to revive, although not in phase with the initial oscillations. This dynamics repeats almost periodically in time, each time making more difficult to distinguish between a collapse and a revival. We now try to interpret this strange behavior.

Let us start with the following observation: Eq. (5.50) is basically a series of cosines, each one characterized by a frequency \( \Omega(n) \) that depends on the number of photons in the field. If the average number of photons is \( \bar{n} \), then it makes sense to say that the dominant Rabi frequency will be \( \Omega(\bar{n}) \), namely

\[
\Omega(\bar{n}) = 2g\sqrt{\bar{n}+1} \simeq 2g\sqrt{\bar{n}},
\]

where the last approximation is valid in the limit \( \bar{n} \gg 1 \), as it is usually the case for a coherent state, where \( \bar{n} \simeq 10^6 \). Since the photon number is not perfectly determined, however, there will be a range of frequencies around \( \Omega(\bar{n}) \) that will contribute sensibly to the dynamics. Calling \( \Delta n \) the uncertainty in the photon number, the relevant frequency range should be between \( \Omega(\bar{n} - \Delta n) \) and \( \Omega(\bar{n} + \Delta n) \). If we want to say something about the dynamics of the system, we therefore have to investigate this frequency range.

**Collapse**  The collapse time \( t_C \) is defined as the instant of time in which the relative dephasing between the various frequency components that are present in the initial expression of \( W(t) \) is such that they all interfere destructively, leaving no atomic inversion, therefore collapsing the dynamics. Using the uncertainty relation existing between time and frequency, we can estimate the collapse time \( t_C \) as follows:

\[
t_C \left[ \Omega(\bar{n} + \Delta n) - \Omega(\bar{n} - \Delta n) \right] \simeq 1.
\]  

Since for a coherent state \( \Delta n = \sqrt{\bar{n}} \), we have that

\[
\omega(\bar{n} \pm \Delta n) \simeq 2g\sqrt{\bar{n} \pm \sqrt{\bar{n}}} \simeq 2g\sqrt{\bar{n}} \left[ 1 \pm \frac{1}{\sqrt{\bar{n}}} \right]^{1/2}
\]

\[
\simeq 2g\sqrt{\bar{n}} \left( 1 \pm \frac{1}{2\sqrt{\bar{n}}} \right) = 2g\sqrt{\bar{n}} \pm g,
\]

and therefore the collapse time is given by

\[
t_C \simeq \frac{1}{2g}.
\]
Interestingly, the collapse time is independent of the average number of photons in the coherent state. This derivation is not fully rigorous, nevertheless it catches the salient features. For a more rigorous derivation, the reader is addressed to Ref. [2].

**Revivals**  We now turn to our attention to determine the revival time $t_R$, corresponding to the time in which all the relevant frequencies are oscillating in phase (or most of them are in the case of a fractional revival), thus reproducing the initial oscillatory pattern. By means of simple considerations based on interference of adjacent frequencies, the revival time can be estimated by means of the following relation:

$$[\Omega(\bar{n} + 1) - \Omega(\bar{n})] t_R = 2\pi k,$$

where $k \in \mathbb{N}$. By using the approximated expressions of $\Omega(\bar{n} + 1)$ and $\Omega(\bar{n})$ given above, we can estimate the revival time to be given by

$$t_R \simeq \left(\frac{2\pi}{g}\right) k\sqrt{\bar{n}}.$$  

(5.56)

In this case, the revival time depends on the average number of photons in the field as a monotonically increasing function. The more photons are in the field, therefore, the more is $t_R$ close to infinity. As for the case of the collapse time, a more rigorous derivation of $t_R$ is given in ref. [2].

### 5.3 Wigner-Weißkopf-Theory [5]

In this section we will study the interaction of a 2 level atom with a multimode field, where the model multimode field is seen as a continuum of states.

In order to study the Wigner-Weißkopf model, we need to introduce the continuous limit to our discrete model of quantization. This can be done by considering two things, which are first taking the limit $V \to \infty$ for the cavity volume and then, accordingly we will need to generalize the definition of $\hat{a}, \hat{a}^\dagger$ to the continuous case.

![Figure 5.3: Geometry of optical cavity (taken from [1])](image)

---

[1]: [Reference needed for the image.]
Since we consider the case $V \to \infty$, the number of allowed $k$-vectors inside the cavity rises, so that we can consider them as continuous. Therefore we will substitute the sum over the $k$-vectors by an integration over the density of states in $k$-space $\mathcal{D}(k)$:

$$
\sum_k \sum_{\lambda=1}^{2} \rightarrow \sum_{\lambda=1}^{2} \int d^3 k \mathcal{D}(k).
$$

(5.57)

In order to obtain $\mathcal{D}(k)$, we consider a quadratic cavity as shown in Fig. 5.3, therefore:

$$
k = \frac{2\pi}{L}(n_x, n_y, n_z),
$$

(5.58)

with $n_i \in \mathbb{N}$, but only one of the $n_i$ can be zero, since the field in the cavity vanishes if two or three $n$'s are zero. For a given combination $(n_x, n_y, n_z)$, there is only one state in the volume $(\frac{2\pi}{L})^3$. Therefore

$$
\mathcal{D}(k) = \frac{V}{(2\pi)^3}.
$$

(5.59)

How do we generalize operators?

With this we get:

$$
\hat{a}_{k\lambda} \rightarrow \sqrt{\Delta \omega} \hat{a}_\lambda(\omega)
$$

(5.60a)

$$
\hat{a}_{k\lambda}^\dagger \rightarrow \sqrt{\Delta \omega} \hat{a}_\lambda^\dagger(\omega)
$$

(5.60b)

$$
\left[\hat{a}_{k\lambda}, \hat{a}_{k\mu}^\dagger\right] = \delta(\omega - \omega') \delta_{\lambda \mu},
$$

(5.60c)

where $\Delta \omega = c\Delta k = \frac{2\pi}{L}$ is a normalization factor introduced to preserve the bosonic nature of the commutator (5.60c). The electric field operator (for a linearly polarized field propagating along $z$-axis) is given by:

$$
\hat{E}_T(z, t) = i \int d\omega \left( \frac{\hbar \omega}{4\pi \varepsilon_0 c A} \right)^{\frac{1}{2}} \left[ \hat{a}(\omega) e^{-i\omega(t-z)} + \hat{a}^\dagger(\omega) e^{i\omega(t-z)} \right].
$$

(5.61)

The magnetic field operator $\hat{B}$ is defined accordingly.

The Model

Our model considers the interaction of a 2-level atom with a single-mode field. For simplicity we only take single photon processes into account. In the beginning of our considerations we assume the atom to be in the excited state and assume no photons in the field, i.e., $|\Phi_2\rangle |0\rangle$. After the interaction we want our atom to be in the ground state (decay by emission of a photon) $|\Phi_1\rangle |\Phi(k)\rangle$, and therefore to have one photon in the field. The emitted photon can have any momentum $k$, meaning that it can be emitted in any direction! This is due to the fact that the atom is inserted in an isotropic vacuum, and there is no preferred direction of emission. Therefore, to
correctly account for all the possibilities, we need to integrate over all the possible values of $k$.

Since both $\{|\Phi_1\rangle, |\Phi_2\rangle\}$ and $|\Phi(k)\rangle$ are orthogonal basis in their respective Hilbert spaces, $|\Psi(0)\rangle = |\Phi_2\rangle |0\rangle$, and we can write the state at time $t$ as superposition of the atomic and field states as follows:

$$
|\Psi(t)\rangle = a(t) \exp\left[ -\frac{iEt}{\hbar} \right] |\Phi_2\rangle + \int d^3k b(k,t) \exp\left[ -\frac{i\varepsilon(k)t}{\hbar} \right] |\Phi(k)\rangle
$$

(5.62)

where $\varepsilon(k)$ is the energy of the photon with momentum $k$. In this case, for the sake of simplicity, we assume to use the ground state as a reference for the energy scale, so that $E = E_2 - E_1 = E_2$. Eq. (5.62) must solve the Schrödinger equation:

$$
i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \left( \hat{H}_0 + \hat{V} \right) |\Psi(t)\rangle,
$$

(5.63)

where $\hat{V}$ is the atom-field interaction, e.g. the dipole Hamiltonian. Calculating the left-hand-side (LHS) and the right-hand-side (RHS) separately, we have:

**LHS**

$$
i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = i\hbar \frac{da(t)}{dt} \exp\left[ -\frac{iEt}{\hbar} \right] |\Phi_2\rangle + E a(t) \exp\left[ -\frac{iEt}{\hbar} \right] |\Phi_1\rangle
$$

$$+
 i\hbar \int d^3k \frac{db(k,t)}{dt} \exp\left[ -\frac{i\varepsilon(k)t}{\hbar} \right] |\Phi(k)\rangle
$$

$$+
 \int d^3k \varepsilon(k) b(k,t) \exp\left[ -\frac{i\varepsilon(k)t}{\hbar} \right] |\Phi(k)\rangle
$$

**RHS**

$$
\left( \hat{H}_0 + \hat{V} \right) |\Psi(t)\rangle = Ea(t) \exp\left[ -\frac{iEt}{\hbar} \right] |\Phi_2\rangle + \int d^3k \varepsilon(k) b(k,t) \exp\left[ -\frac{i\varepsilon(k)t}{\hbar} \right] |\Phi(k)\rangle
$$

$$+
 a(t) \exp\left[ -\frac{iEt}{\hbar} \right] \hat{V} |\Phi_2\rangle + \int d^3k b(k,t) \exp\left[ -\frac{i\varepsilon(k)t}{\hbar} \right] \hat{V} |\Phi(k)\rangle
$$

Remember $\hat{V}$ is a dipole-line separator given by

$$
\langle \Phi_k | \hat{V} | \Phi_k \rangle = 0,
$$

(5.64)

and the photon states are orthogonal, i.e.,

$$
\langle \Phi(q) | \Phi(k) \rangle = \delta(q - k).
$$

(5.65)

If we project the Schrödinger eq. (5.63) onto $|\Phi_2\rangle$ and $|\Phi(k)\rangle$:

$$
i\hbar \frac{da(t)}{dt} = \int d^3k b(k,t) M(k) e^{-i\omega(k)t},
$$

(5.66a)
\[
i\hbar \frac{db(k,t)}{dt} = a(t) M^*(k) e^{i\omega(k)t}, \tag{5.66b}
\]

where \( h \omega(k) = \varepsilon(k) - E \) is the transmission energy, \( M(k) = \langle \Phi_2 | \hat{V} | \Phi(k) \rangle \) gives the matrix element and with the initial conditions \( a(0) = 1 \), \( b(k,0) = 0 \).

Eq. (5.66b) can be formally solved by multiplying by \( dt \) and integrating, namely

\[
\int_0^t db(k,t') = \frac{M^*(k)}{i\hbar} \int_0^t dt' a(t') e^{i\omega(k)t'},
\]

thus obtaining

\[
b(k,t) = \frac{M^*(k)}{i\hbar} \int_0^t dt' a(t') e^{i\omega(k)t'}. \tag{5.67}
\]

Substituting the expression of \( b(k,t) \) by Eq. (5.67) we obtain for Eq. (5.66a)

\[
\frac{da(t)}{dt} = \frac{1}{i\hbar} \int d^3 k \frac{|M(k)|^2}{i\hbar} e^{i\omega(k)t} \int_0^t dt' a(t') e^{i\omega(k)t'},
\]

i.e.,

\[
\frac{da(t)}{dt} = -\frac{1}{\hbar^2} \int d^3 k |M(k)|^2 e^{i\omega(k)t} \int_0^t dt' a(t') e^{i\omega(k)t'}. \tag{5.68}
\]

The last term of Eq. (5.68) makes the problem non-Markovian, i.e. the system has memory and therefore depends on its past history. We need to make it Markovian, i.e. \( a(t_2 > t_1) \) does not have to depend from all \( a(t) \) with \( t \in [0,t_1] \). Here we use a little trick: we will multiply the equation by \( e^{-\lambda t} \) and integrate in time. We then have from Eq. (5.68)

**LHS**

\[
\int_0^\infty dt \frac{da(t)}{dt} e^{-\lambda t} = \int_0^\infty dt \left\{ \frac{d}{dt} \left[ a(t) e^{-\lambda t} \right] + \lambda a(t) e^{-\lambda t} \right\}
\]

\[
= a(t) e^{-\lambda t} \big|_0^\infty + \lambda \int_0^\infty dt a(t) e^{-\lambda t}
\]

\[
= \lambda \int_0^\infty dt a(t) e^{-\lambda t} - 1.
\]
Here we used the fact that the underlined integral can be solved in two ways (see Fig. 5.4), i.e., whether considering $t$ as fixed and integrating along a vertical strip, \[ \int_{0}^{t} dt \int_{t'}^{\infty} dt' \], which corresponds to \[ \int_{0}^{\infty} dt \int_{0}^{t} dt' \], or integration along a horizontal strip (i.e. \[ \int_{t'}^{t} dt' \int_{0}^{\infty} dt \]) where we first integrate over \[ \int_{0}^{t} dt' \], which corresponds to switching the integrations in Eq. (5.68), i.e., \[ \int_{0}^{t} dt' \int_{t'}^{\infty} dt \].

In this case, we use the second possibility, to integrate the RHS of Eq. (5.68) and then obtaining

\[
\int_{0}^{\infty} dt e^{-\lambda t} a(t) = \left[ \lambda + \frac{1}{\hbar^2} \int d^3k \frac{|M(k)|^2}{\lambda + i\omega(k)} \right]^{-1}. \tag{5.69}
\]

**Note:** If we go back to Eq. (5.68) and assume $|M(k)|^2$ to be independent of $k$ we
obtain:

\[
\frac{da(t)}{dt} = -\frac{1}{\hbar^2} \int d^3k \ |M(k)|^2 e^{-i\omega(k)t} \int_0^t dt' a(t') e^{i\omega(k)t'}
\]

(5.70)

\[
= -\frac{|M|^2}{\hbar^2} \int d^3k e^{-i\omega(k)t} \int_0^t dt' a(t') e^{i\omega(k)t'}
\]

(5.71)

\[
= -\frac{|M|^2}{\hbar^2} \int_0^t dt' a(t') \int d^3k e^{-i\omega(k)(t-t')} \sim \delta(t-t') = -\frac{|M|^2}{\hbar^2} a(t),
\]

(5.72)

and therefore:

\[
a(t) = a(0) e^{-\gamma t},
\]

(5.73)

with \(\gamma = \frac{|M|^2}{\hbar^2}\). This solution is consistent with the assumption of having an isotropic vacuum. In this case, in fact, there is no reason why the emitted photon should prefer one \(k\) vector to the other one, as they are all equally probable. Therefore, the matrix element \(M(k)\) cannot (by symmetry considerations) depend on \(k\). Now let us assume that this would still be a good solution even for the case that \(M(k)\) depends on \(k\) and solve the problem for \(\lambda \to 0\).

Keeping \(a(t) = e^{-zt}\) as Ansatz, we then have:

\[
\int_0^\infty dt e^{-\lambda t} e^{-zt} = -\frac{1}{z + \lambda} e^{-(z+\lambda)t} \bigg|_0^\infty = \frac{1}{z + \lambda},
\]

(5.74)

and therefore Eq. (5.69) becomes

\[
\frac{1}{z + \lambda} = \left[ \lambda - i \frac{\hbar^2}{2} \int d^3k \frac{|M(k)|^2}{\omega(k) - i\lambda} \right]^{-1},
\]

(5.75)

where we have used the fact that

\[
\frac{1}{\hbar^2} \frac{1}{\lambda + i\omega} = \frac{1}{\hbar^2} \frac{1}{i(\omega - i\lambda)} = -\frac{i}{\hbar^2} \frac{1}{\omega - i\lambda}.
\]

As \(\lambda\) is an artificial parameter inserted for facilitating the solution of Eq. (5.68), the real solution must not depend on it. To do so, we then take the limit of (5.75) for \(\lambda \to 0\), thus obtaining

\[
z = \lim_{\lambda \to 0^+} \left\{ -\frac{i}{\hbar^2} \int d^3k \frac{|M(k)|^2}{\omega(k) - i\lambda} \right\}
\]

(5.76)
By using
\[ \lim_{\lambda \to 0^+} \frac{\lambda}{\omega^2(k) + \lambda^2} = \pi \delta(\omega(k)), \quad (5.77a) \]
and
\[ \delta(ax) = \frac{1}{|a|} \delta(x). \quad (5.77b) \]

Eq. (5.76) becomes
\[ z = -i \frac{\hbar}{\pi} \int d^3 k \frac{|M(k)|^2}{\omega(k)} + \frac{\pi}{\hbar} \int d^3 k |M(k)|^2 \delta(h\omega(k)). \quad (5.78) \]

In order to better understand this result, we define
\[ E' = E + \int d^3 k \left| \langle \Phi_1 | \hat{V} | \Phi(k) \rangle \right|^2, \quad (5.79a) \]
\[ \gamma = \frac{2\pi}{\hbar} \int d^3 k \left| \langle \Phi_1 | \hat{V} | \Phi(k) \rangle \right|^2 \delta(\epsilon(k) - E), \quad (5.79b) \]
so that
\[ z = iE'/\hbar + \pi \gamma/(2\hbar). \]

Then, the solution for \( a(t) \) reads as follows:
\[ a(t) = e^{-iE't/\hbar} e^{-\gamma t}, \quad (5.80) \]

and the occupation probability can be simply calculated as \( |a(t)|^2 \), i.e.,
\[ P_l(t) = |a(t)|^2 = e^{-\gamma t}. \quad (5.81) \]

The excited state of the atom decays exponentially in time due to the interaction with the electromagnetic field in the vacuum state.

Let us then now calculate, what is the probability that at time \( t \to \infty \) the system is in the state \( |\Phi(k)\rangle \) (atom decayed and emitted a photon).

To do that, we calculate \( b(k, t) \) using the above result \( a(t) = e^{-\gamma t} \), thus obtaining
\[ b(k, \infty) = \left[ \frac{M^* (k)}{i\hbar} \right] \int_0^\infty dt' e^{-z - i\omega(k)t'} \]
\[ = \left[ \frac{M^* (k)}{i\hbar} \right] \left[ \frac{\pi}{\hbar} \int d^3 k' |M(k')|^2 \delta(h\omega(k)) - i\Delta \right]^{-1}, \quad (5.82) \]
where
\[ \Delta = \frac{\epsilon(k) - E}{\hbar} - \frac{1}{\hbar} \int d^3 k' |M(k')|^2 \frac{E - \epsilon(k')}{E - \epsilon(k)}. \quad (5.83) \]
Rearranging the terms gives the following final result:

\[
b(k, \infty) = \frac{M^*(k)}{\varepsilon(k') - E - \frac{i}{\hbar} \int d^3k' \frac{|M(k')|^2}{E - \varepsilon(k')} + \frac{i\hbar \gamma}{2}, \tag{5.84}
\]

and therefore the probability that a photon with momentum \( k \) has been emitted is given by

\[
P_k = |b(k, \infty)|^2 = \frac{|M(k)|^2}{\varepsilon(k') - E + \left(\frac{\hbar \gamma}{2}\right)^2}. \tag{5.85}
\]

Usually, for simplicity, in our models we assume that the transition of electrons between energetic level has a delta-like resonance function, i.e., the emitted photon is monochromatic. But as seen in Eq. (5.85) this is not true, because the transition probability has a Lorentzian shape with non-zero bandwidth. Therefore the emitted photons show a spectral broadening, which we call natural linewidth. The natural linewidth gives the lower limit to the spectral bandwidth of our photon.

6 Quantum Mechanics of Beam Splitters

6.1 Classical theory

![Schematic sketch of a classical beam splitter with incoming and outgoing fields](image)

Figure 6.1: Schematic sketch of a classical beam splitter with incoming and outgoing fields

The beam splitter in Fig. 6.1 is assumed to be ideal, i.e., it has two different surfaces and its thickness can be assumed to be small enough to be neglected. The reflection and transmission coefficients are \((r, t)\) and \((r', t')\) for the first (solid line in Fig. 6.1) and second (dashed line in Fig. 6.1) surface, respectively. The incoming and
outgoing fields can be then written as

\[ E_{IN}(r,t) = \frac{1}{2} e^{-i\omega t} \left[ a_1 e^{i k_1 \cdot r} + a_2 e^{i k_2 \cdot r} \right], \quad (6.1a) \]

\[ E_{OUT}(r,t) = \frac{1}{2} e^{-i\omega t} \left[ a_3 e^{i k_3 \cdot r} + a_4 e^{i k_4 \cdot r} \right], \quad (6.1b) \]

Where \( a_i \) are the amplitudes of the ingoing \((a_1, a_2)\) and outgoing \((a_3, a_4)\) fields, respectively. We can write this in a matrix form, obtaining

\[
\begin{pmatrix}
  a_4 \\
  a_3
\end{pmatrix} =
\begin{pmatrix}
  t & r' \\
  r & t'
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix} = B
\begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix},
\]

where due to energy conservation \( |a_1|^2 + |a_2|^2 = |a_3|^2 + |a_4|^2 \).

If we define:

\[ \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (6.3a) \]

and

\[ \mathbf{u} = B \mathbf{v}, \quad (6.3b) \]

then the total energy reads

\[ \mathcal{E}_{in} = |E_{in}|^2 = (\mathbf{v}, \mathbf{v}), \quad (6.4) \]

and

\[ \mathcal{E}_{out} = (\mathbf{u}, \mathbf{u}) = (B \mathbf{v}, B \mathbf{v}) = (\mathbf{v}, B \dagger B \mathbf{v}) = \mathcal{E}_{in} = (\mathbf{v}, \mathbf{v}), \quad (6.5) \]

where we used \((AB, C) = (A, B \dagger C)\). We get that \( \mathcal{E}_{out} = \mathcal{E}_{in} \) if \( B \dagger B = 1 \).

What does this mean? If \( B \dagger B = 1 \rightarrow B \dagger = B^{-1} \), then

\[ B = \begin{pmatrix}
  t & r' \\
  r & t'
\end{pmatrix} \rightarrow B \dagger = \begin{pmatrix}
  t^* & r'^* \\
  r'^* & t^*
\end{pmatrix}, \]

and

\[ B^{-1} = \frac{1}{tt' - rr'} \begin{pmatrix}
  t' & -r' \\
  -r & t
\end{pmatrix}, \]

where \( tt' - rr' \) is just a phase factor. Therefore

\[ B \dagger B = 1 \rightarrow \begin{cases} 
  t' = t^*, \\
  r' = -r^*,
\end{cases} \]

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and so the beam splitter can be represented by the following matrix

\[
B = \begin{pmatrix} t & -r^* \\ r^* & t^* \end{pmatrix}.
\] (6.6)

Moreover it follows that:

\[
\frac{t'}{r'} = -\left( \frac{t}{r} \right)^*.
\] (6.7)

This result allows us to obtain the following relation for the phases of the transmission and reflexion coefficients:

\[
(\Phi_{\nu} - \Phi_{\nu'}) + (\Phi_t - \Phi_r) = \pm \pi + 2n\pi.
\]

If we choose

\[
\Phi_t - \Phi_r = \Phi_{\nu'} - \Phi_{\nu} = \frac{\pi}{2},
\]

this leads to

\[
B = \begin{pmatrix} t & i r \\ i r & t \end{pmatrix}.
\] (6.8)

**Note:**

1. Energy conservation is fulfilled:

\[
|t|^2 + |r|^2 = 1
\]

2. because of the first point we can write \(t = \cos \vartheta, r = \sin \vartheta\), so that \(|t|^2 + |r|^2 = \cos^2 \vartheta + \sin^2 \vartheta = 1\) and so

\[
B = \begin{pmatrix} \cos \vartheta & i \sin \vartheta \\ i \sin \vartheta & \cos \vartheta \end{pmatrix}.
\]
6.2 Quantum Theory

Let us consider a beam splitter with very high transmission ($\varepsilon \ll 1$). We want an operator representation for a beam splitter from a quantum mechanical point of view, therefore the operator must be unitary and linear in $\hat{a}^\dagger, \hat{a}$. Hence let us look for an exponential operator. We obtain the simple result:

$$\begin{pmatrix} \hat{a}_4 \\ \hat{a}_3 \end{pmatrix} = B \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix},$$

(6.9)

Proof:

We want a unitary operator $\hat{U}$ such that

$$\hat{U}\hat{a}_1\hat{U} = \hat{a}_4,$$

(6.10a)

$$\hat{U}\hat{a}_2\hat{U} = \hat{a}_3.$$

(6.10b)

1Note that the transformation between the input mode operators $\hat{a}_1$ and $\hat{a}_2$ and the correspondent output mode operators $\hat{a}_3$ and $\hat{a}_4$ implemented by the unitary operator $\hat{U}$ is a particular case of a more general class of transformations, namely the so-called Bogoliubov transformations. Bogoliubov transformations are unitary transformations from a unitary representation of a commutation (or anti-commutation) algebra into another, equivalent and unitary representation. They are often employed in theoretical physics and condensed matter theory to diagonalize Hamiltonians, leading to the steady-state solutions of the correspondent Schrödinger equations. Moreover, Bogoliubov transformation are an essential ingredient to understand the Unruh effect and the Hawking radiation. The interested reader can consult Refs. [6, 7, 8] for more detailed informations about the use of Bogolyubov transformations in quantum field theory and condensed matter theory.
Since \( \hat{U} \) is unitary, we can write it in the form of an exponential operator, namely

\[
\hat{U} = e^{-i\hat{H}},
\]

with

\[
\hat{H}^\dagger = \hat{H},
\]

so that

\[
|\text{out}\rangle = \hat{U} |\text{in}\rangle = e^{-i\hat{H}} |\text{in}\rangle. \tag{6.11}
\]

If we assume low reflectivity \((\varepsilon \ll 1)\), then \(\hat{H} = \hat{H}(\varepsilon)\) can be expanded in a Taylor series with respect to \(\varepsilon\), i.e.,

\[
\hat{H}(\varepsilon) = \hat{H}_0 + \varepsilon \hat{H}' + O(\varepsilon^2).
\]

Plugging \(\hat{H}(\varepsilon)\) in (6.11) and Taylor expanding of the exponential we obtain:

\[
|\text{out}\rangle = e^{-i\hat{H}} |\text{in}\rangle = e^{-i[\hat{H}_0 + \varepsilon \hat{H}']} |\text{in}\rangle \approx \left(1 - i\varepsilon \hat{H}'\right) |\text{in}\rangle = |k_1\rangle - i\varepsilon \hat{H}' |k_1\rangle \propto |k_1\rangle + \varepsilon |k_2\rangle. \tag{6.12}
\]

Note that

\[
\hat{H}' |k_1\rangle \propto |k_2\rangle
\]

which leads to

\[
\langle k_2 | \hat{H}' |k_1\rangle = c - \text{number} \tag{6.13}
\]

How can we find an expression for \(\hat{H}'\)? Given Eq. (6.13), we can say that

\[
\hat{H}' = \hat{a}_2^\dagger \hat{a}_1 + \hat{Q},
\]

with

\[
\langle k_2 | \hat{Q} |k_1\rangle = 0.
\]

From \((\hat{H}') = (\hat{H}'^\dagger)^\dagger\), follows that

\[
\hat{a}_2^\dagger \hat{a}_1 + \hat{Q} = \hat{a}_1^\dagger \hat{a}_2 + \hat{Q}^\dagger,
\]

and therefore it is not difficult to prove that

\[
\hat{Q} = \hat{a}_1^\dagger \hat{a}_2.
\]

Therefore we obtain:

\[
\hat{H}' = \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1, \tag{6.14}
\]
Quantum Mechanics of Beam Splitters

which in general reads

\[ \hat{H}' = g^* \hat{a}^\dagger_1 \hat{a}_2 + g \hat{a}^\dagger_2 \hat{a}_1. \]  \hfill (6.15)

The Beam Splitter operator can be then written as follows:

\[ \hat{U} = e^{i \varepsilon (g^* \hat{a}^\dagger_2 \hat{a}_1 + g \hat{a}^\dagger_1 \hat{a}_2)}. \]  \hfill (6.16)

Now we will prove, that the matrix representation of the beam splitter operator introduced above is given by \( \hat{B} \), therefore we introduce:

\[ x = \left( \begin{array}{c} \hat{a}_1 \\ \hat{a}_2 \end{array} \right), \]  \hfill (6.17)

and build the following scalar product

\[ (x, M x) = x^\dagger M x = m_{11} \hat{a}_1^\dagger \hat{a}_1 + m_{12} \hat{a}_1^\dagger \hat{a}_2 + m_{21} \hat{a}_2^\dagger \hat{a}_1 + m_{22} \hat{a}_2^\dagger \hat{a}_2 = \hat{B}, \]  \hfill (6.18)

with \( M = M^\dagger \). Let us now assume to introduce an operator \( \hat{U} (\lambda) \) such that

\[ \hat{U} (\lambda) = e^{i \lambda \hat{B}}. \]  \hfill (6.19)

In this case \( \hat{U} (1) = e^{i \hat{B}} \) and calculate

\[ \hat{U} x \hat{U}^\dagger = e^{i \lambda \hat{B}} x e^{-i \lambda \hat{B}} \equiv \hat{f} (\lambda). \]

By using the Heisenberg equation we can see how \( \hat{f} (\lambda) \) evolves with respect to \( \lambda \), i.e.,

\[ i \frac{d \hat{f}}{d \lambda} = i \left[ \hat{B}, \hat{f} (\lambda) \right], \]  \hfill (6.20)

with \( \hat{f} (0) = \hat{x} \). Calculating the RHS gives

\[ \left[ \hat{B}, \hat{f} (\lambda) \right] = \left[ \hat{B}, e^{i \lambda \hat{B}} \hat{x} e^{-i \lambda \hat{B}} \right] = \hat{B} e^{i \lambda \hat{B}} \hat{x} e^{-i \lambda \hat{B}} - e^{i \lambda \hat{B}} \hat{x} e^{-i \lambda \hat{B}} \hat{B} = e^{i \lambda \hat{B}} \left[ \hat{B}, \hat{x} \right] e^{-i \lambda \hat{B}} \]  \hfill (6.21)

(note that \( \hat{B} \) and \( e^{\pm i \lambda \hat{B}} \) commute). Computing the matrix element of the commutator, with \( \hat{B} = m_{ij} \hat{a}_i^\dagger \hat{a}_j \) gives

\[ \left[ \hat{B}, \hat{x} \right] = m_{ij} \hat{a}_i^\dagger \hat{a}_j = m_{ij} \left[ \hat{a}_i^\dagger, \hat{a}_j \right] = m_{ij} \{ \hat{a}_i^\dagger [\hat{a}_j, \hat{a}_k] + [\hat{a}_i^\dagger, \hat{a}_k] \hat{a}_j \} = -m_{ij} \delta_{ik} \hat{a}_j = -m_{kj} \hat{a}_j = (M \hat{x})_k. \]  \hfill (6.22)
By then generalizing the previous result we get:

\[
\begin{align*}
\left[ \hat{B}, \hat{f} (\lambda) \right] &= -e^{i\lambda \hat{B}} M \hat{x} e^{-i\lambda \hat{B}} \\
&= -M e^{i\lambda \hat{B}} \hat{x} e^{-i\lambda \hat{B}} \\
&= -M \hat{f} (\lambda)
\end{align*}
\] (6.23)

(note that \( M \) is a c-number matrix), i.e., it is not an operator anymore. Eq. (6.20) now becomes

\[
\frac{d \hat{f}}{d\lambda} = -i M \hat{f},
\]

whose solution is given by

\[
\hat{f} (\lambda) = e^{-i\lambda M} \hat{x}.
\] (6.24)

Observe that \( \hat{U}^\dagger \hat{x} \hat{U} \) can be then represented in two equivalent ways, i.e.,

\[
\hat{U}^\dagger \hat{x} \hat{U} = e^{i(g^* \hat{a}_2^\dagger \hat{a}_1 + ga_1^\dagger a_2)} \hat{x} e^{-i(g a_1^\dagger a_1 + g^* \hat{a}_2^\dagger \hat{a}_2)},
\] (6.25a)

and

\[
\hat{U}^\dagger \hat{x} \hat{U} = e^{i\lambda \hat{B}} \hat{x} e^{-i\lambda \hat{B}} = e^{im_{ij} \hat{a}_i^\dagger \hat{a}_j} M e^{-im_{ij} \hat{a}_i^\dagger \hat{a}_j}.
\] (6.25b)

Comparing these two equivalent representations, i.e., the RHS of (6.25a) with the RHS of (6.25b) we get

\[
M = \begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}.
\] (6.26)

Exponentiating \( M \) we have the following result

\[
e^{-iM} = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} M^k = \begin{pmatrix} \cos |g| & -i \frac{g^*}{|g|} \sin |g| \\ -i \frac{g}{|g|} \sin |g| & \cos |g| \end{pmatrix}.
\] (6.27)

If the phases \( \frac{g}{|g|} \) and \( \frac{g^*}{|g|} \) are properly chosen, then

\[
\begin{pmatrix} \cos |g| & -i \frac{g^*}{|g|} \sin |g| \\ -i \frac{g}{|g|} \sin |g| & \cos |g| \end{pmatrix} \rightarrow \begin{pmatrix} t & ir \\ ir & t \end{pmatrix},
\]

therefore

\[
t = \cos |g|,
\] (6.28a)

and

\[
r = -i e^{i\phi} \sin |g|.
\] (6.28b)
6.3 Classical VS Quantum Beam Splitter

6.3.1 Classical BS

We consider a classical beam splitter, where we have a field in only one arm. Therefore the beam splitter equation reads

\[
\begin{pmatrix}
a_4 \\
a_3
\end{pmatrix} = \begin{pmatrix} t & i r \\
i r & t \end{pmatrix} \begin{pmatrix} a_1 \\ 0 \end{pmatrix}. \tag{6.29}
\]

The incoming and outgoing energy then are given by

\[
\mathcal{E}_{in} = |v_1|^2, \tag{6.30a}
\]

and

\[
\mathcal{E}_{out} = |a_3|^2 + |a_4|^2 = |irv_1|^2 + |ta_1|^2 = \left( |r|^2 + |t|^2 \right) |v_1|^2 = |a_1|^2. \tag{6.30b}
\]

As one can expect the energy conservation is fulfilled in this case.

6.3.2 Quantum BS

As in the case of a classical beam splitter we want to study the case of a quantum beam splitter where we have a field only in one input. The beam splitter equation for the quantum case then reads

\[
\begin{pmatrix}
\hat{a}_4 \\
\hat{a}_3
\end{pmatrix} = \begin{pmatrix} t & i r \\
i r & t \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ 0 \end{pmatrix}. \tag{6.31}
\]

Since the beam splitter is a passive device, the nature of the incoming radiation must not be changed. Therefore, if the bosonic commutation relations (see Eq. (6.32)) hold for the incoming field, they must also hold for the outgoing fields.

\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \tag{6.32a}
\]

\[
[\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]. \tag{6.32b}
\]

In order to see if above statement is valid we need to compute \([\hat{a}_3, \hat{a}_3^\dagger]\) (where we expect result=1), \([\hat{a}_4, \hat{a}_4^\dagger]\) (=1) and \([\hat{a}_3, \hat{a}_4^\dagger]\) (=0)

\[
[\hat{a}_3, \hat{a}_3^\dagger] = [ir\hat{a}_1, -ir^*\hat{a}_1^\dagger] = |r|^2 [\hat{a}_1, \hat{a}_1^\dagger] = |r|^2 \neq 1 \tag{6.33a}
\]
\[
\begin{align*}
\hat{a}_4 \hat{a}_4^\dagger &= [t \hat{a}_1, t^* \hat{a}_1^\dagger] = |t|^2 [\hat{a}_1, \hat{a}_1^\dagger] = |t|^2 \neq 1 \quad (6.33b) \\
\hat{a}_3 \hat{a}_4^2 &= [ir \hat{a}_1, -ir^* \hat{a}_1^\dagger] = ir t^* [\hat{a}_1, \hat{a}_1^\dagger] = ir t^* \neq 0 \quad (6.33c)
\end{align*}
\]

Why are these results different from our expectations? In order to understand why the results are not the expected ones let us consider one little (subtle) detail (see Fig. 6.3):

Figure 6.3: Comparison classical vs quantum BS

As we see from Fig. 6.3 we need to account for the presence of the quantum vacuum in the second arm of the BS. We indicate this by assigning an operator \( \hat{a}_0 \) on the other input port of the BS. In this case Eq. (6.31) becomes

\[
\begin{pmatrix}
\hat{a}_4 \\
\hat{a}_3
\end{pmatrix}
= 
\begin{pmatrix}
t & ir \\
ir & t
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_0
\end{pmatrix},
\]

if we now calculate for example \( [\hat{a}_3, \hat{a}_3^\dagger] \), we get

\[
[\hat{a}_3, \hat{a}_3^\dagger] = [ir \hat{a}_1 + t \hat{a}_0, -ir^* \hat{a}_1^\dagger + t^* \hat{a}_0^\dagger] \\
= |r|^2 [\hat{a}_1, \hat{a}_1^\dagger] + ir t^* [\hat{a}_1, \hat{a}_0^\dagger] - irt [\hat{a}_0, \hat{a}_1^\dagger] + |t|^2 [\hat{a}_0, \hat{a}_0^\dagger] \\
= |r|^2 + |t|^2 = 1,
\]

which is the expected result. The other commutator follows analogously.
6.4 Effects of Vacuum in BS: Coherent State Input

We now want to ask ourselves if taking into account the presence of the vacuum state in the unused port of the BS has some sort of consequence or not. To this aim, let us first consider the case of a coherent state as input, and assume, for simplicity, to consider a 50% BS, i.e., \( t = r = 1/\sqrt{2} \). The input state is then given by (remember displacement operator, see 3.52):

\[
|\text{in}\rangle = |\alpha\rangle_1 |0\rangle_2 = \hat{D}(\alpha)|0\rangle_1 |0\rangle_2 = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle_1 |0\rangle_2.
\]  

(6.35)

And, according to Eq. (6.34)

\[
\begin{pmatrix}
\hat{a}_4 \\
\hat{a}_3
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\
-i & 1
\end{pmatrix} \begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix}.
\]

Since we want to calculate how the input state gets transformed by the BS, it is more useful to express the input operators \( \hat{a}_1 \) and \( \hat{a}_2 \) as a function of the output operators \( (\hat{a}_3, \hat{a}_4) \). This is obtained by inverting the above equation, thus obtaining

\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\
i & 1
\end{pmatrix} \begin{pmatrix}
\hat{a}_3 \\
\hat{a}_4
\end{pmatrix},
\]

(6.36)

we then have that the input state \( |\text{in}\rangle \) transforms to the following output state:

\[
|\text{out}\rangle = \hat{D}'(\alpha)|0\rangle_3 |0\rangle_4 = e^{\frac{\alpha}{\sqrt{2}}(\hat{a}_3^\dagger+i\hat{a}_4^\dagger)} - e^{\frac{-\alpha}{\sqrt{2}}(\hat{a}_4^\dagger-i\hat{a}_3^\dagger)} |0\rangle_3 |0\rangle_4,
\]

(6.37)

Moreover, since \( \hat{a}_4 \) and \( \hat{a}_3 \) commute

\[
e^{\frac{\alpha}{\sqrt{2}}(\hat{a}_3^\dagger+i\hat{a}_4^\dagger)} - e^{\frac{-\alpha}{\sqrt{2}}(\hat{a}_4^\dagger-i\hat{a}_3^\dagger)} = \exp \left[ \frac{i\alpha \hat{a}_3^\dagger}{\sqrt{2}} - \frac{i\alpha^* \hat{a}_3^\dagger}{\sqrt{2}} \right] \exp \left[ \frac{\alpha \hat{a}_4^\dagger}{\sqrt{2}} - \frac{\alpha^* \hat{a}_4^\dagger}{\sqrt{2}} \right]
\]

\[
= \hat{D}_3 \left( \frac{i\alpha}{\sqrt{2}} \right) \hat{D}_4 \left( \frac{\alpha}{\sqrt{2}} \right).
\]

(6.38)

As a result we have that the input coherent state is split (50%) into two coherent states. The average number of photons per output port is therefore 50% of the initial one, since coherent states contain the quantum noise, this corresponds to the classical case.
6.5 Effects of Vacuum in BS: Single Photon Input

We now consider the case of a single photon in a Fock state impinging on a 50% BS. Again, \( r = t = \frac{1}{\sqrt{2}} \), and we have

\[
|1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle_1 |0\rangle_2 \xrightarrow{\text{BS}} \frac{1}{\sqrt{2}} (\hat{a}_4^\dagger + i\hat{a}_3^\dagger) |0\rangle_3 |0\rangle_4 \\
= \frac{1}{\sqrt{2}} (i|1\rangle_3 |0\rangle_4 + |0\rangle_3 |1\rangle_4).
\]

We have therefore:

\[
|1\rangle_1 |0\rangle_2 \rightarrow \frac{1}{\sqrt{2}} (i|1\rangle_3 |0\rangle_4 + |0\rangle_3 |1\rangle_4).
\]  

(6.40)

This means that the output state of the BS is, because of the impact of the vacuum state in the second port of the input, an entangled state, where we can not tell anymore, where the photon is.

6.6 Hong-Ou-Mandel Effect [9]

Consider a 50% BS with two photons as input, one single photon from the field in first arm and one single photon from the vacuum in second port. What will be our output? Let us have a look at the possibilities how two photons would behave the classical case:

![Classical prediction of the behaviour of two photons in a BS](image)

Figure 6.4: Classical prediction of the behaviour of two photons in a BS

In order to check if our expectations are correct, let us do some QM calculations.
First of all let us rewrite the input operators as a function of the output ones, namely

\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -i \\
i & 1
\end{pmatrix}
\begin{pmatrix}
\hat{a}_4 \\
\hat{a}_3
\end{pmatrix}
\]

and similarly, for the hermite conjugate

\[
\begin{pmatrix}
\hat{a}^\dagger_1 \\
\hat{a}^\dagger_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & i \\
i & 1
\end{pmatrix}
\begin{pmatrix}
\hat{a}^\dagger_4 \\
\hat{a}^\dagger_3
\end{pmatrix}.
\]

Since the incoming field is given by:

\[
|\text{in}\rangle = |1\rangle_1 |1\rangle_2 = \hat{a}^\dagger_1 \hat{a}^\dagger_2 |0\rangle,
\]

the outgoing field is then given by

\[
|\text{out}\rangle = \left[ \frac{1}{\sqrt{2}} \left( \hat{a}^\dagger_4 + i \hat{a}^\dagger_3 \right) \right] \left[ \frac{1}{\sqrt{2}} \left( i \hat{a}^\dagger_4 + \hat{a}^\dagger_3 \right) \right] |0\rangle \\
= \frac{1}{2} \left[ i \left( \hat{a}^\dagger_4 \right)^2 + \hat{a}^\dagger_4 \hat{a}^\dagger_3 - \hat{a}^\dagger_3 \hat{a}^\dagger_4 + i \left( \hat{a}^\dagger_3 \right)^2 \right] |0\rangle \\
= \frac{1}{\sqrt{2}} \left[ i \frac{\left( \hat{a}^\dagger_4 \right)^2}{\sqrt{2}} + i \frac{\left( \hat{a}^\dagger_3 \right)^2}{\sqrt{2}} \right] |0\rangle \\
= \frac{1}{\sqrt{2}} \left( |2\rangle_4 |0\rangle_3 + |0\rangle_4 |2\rangle_3 \right).
\]

\[\text{(6.43)}\]

---

**Figure 6.5: Schematic sketch of a HOM interferometer**

Fig. 6.5 shows schematically how a HOM interferometer works. Here a nonlinear process (PDC-cell) is used to generate single photon states. The BS works as seen
above. If we slightly move the BS, we can introduce a delay between the two counts. The detected signal is illustrated in Fig. 6.6.

\[ R_{\text{coincidence}} \approx 1 - \exp \left( -\Delta \omega^2 (\tau_s - \tau_i)^2 \right) , \quad (6.44) \]

where \(-\Delta \omega^2\) is the bandwidth of the photons (non-monochromatic signal/idler) and \(\tau_s - \tau_i\) is proportional to the distance difference that the two photons travelled. If \(\tau_s - \tau_i \gg \tau_{\text{corr}}\) we obtain a maximum, i.e. the photons are not impinging simultaneously on the BS, which shows classical behaviour (simultaneous clicks possible). But for \(\tau_s - \tau_i = 0\) we obtain that \(R_{\text{coincidence}} = 0\), i.e. the photons are impinging simultaneously on the BS, therefore we observe bunching (no simultaneous click possible) and hence, the coincidence is zero, as can be seen.

7 Quantum Theory of Photodetection

7.1 Direct Detection

Number of photons that arrive at the detector during the integration time \(t\) is given by

\[ \tilde{M}(t, \tau) = \int_t^{t+\tau} dt' \hat{a}^\dagger (t') \hat{a}(t') \quad (7.1) \]

just like the time averaged classical intensity (average over integration time). Let us assume, that the light is nearly monochromatic and that \(T < \infty\). This is the ideal
Quantum Theory of Photodetection

In reality we have no detector with a 100% conversion efficiency (photons may fail to trigger ionization events, dead time masks subsequent ionization events). Dead time is the time that the photo-tube needs to recover, therefore we have a certain quantum efficiency $\eta < 1$. Therefore we can model a real (imperfect/inefficient) photo-detector as shown in fig. 7.1.

![Figure 7.1: Schematic sketch of the model of a real detector](image)

Fig. 7.1 shows the actual model of a real detector. The BS models the inefficiency and $\hat{v}(t)$ accounts for the quantum noise. The transmittance and reflectivity of the BS are given by:

$$T = \sqrt{\eta} \quad (7.2a)$$

$$R = i \sqrt{1 - \eta} \quad (7.2b)$$

- The output on arm 3 is lost (detector inefficiency)
- the output on arm 4 enters a perfect detector
- $\hat{v}(t)$ is a continuous vacuum

We have:

$$\hat{d}(t) = \sqrt{\eta} \hat{a}^\dagger(t) + i \sqrt{1 - \eta} \hat{v}(t) \quad (7.3)$$
The photo-count operator $\hat{M}$ is now defined as

$$\hat{M}_D (t, \tau) = \int_t^{t+\tau} dt' \hat{d} (t') \hat{d} (t') .$$  

(7.4)

Mean Photocount:

$$\langle m \rangle \equiv \langle \hat{M}_D (t, \tau) \rangle = \langle \int_t^{t+\tau} dt' \hat{d} (t') \hat{d} (t') \rangle $$

$$= \langle \int_t^{t+\tau} dt' \left[ \sqrt{\eta} \hat{a} (t') - i \sqrt{1-\eta} \hat{v} (t') \right] \left[ \sqrt{\eta} \hat{a} (t') + i \sqrt{1-\eta} \hat{v} (t') \right] \rangle $$

$$= \eta \langle \hat{M} (t, \tau) \rangle + i \sqrt{\eta} (1-\eta) \langle \int_t^{t+\tau} dt' \hat{a} (t') \hat{v} (t') \rangle $$

and therefore:

$$\langle m \rangle \equiv \eta \langle \hat{M} (t, \tau) \rangle$$  

(7.5)

Second Moment:

$$\langle m(m - 1) \rangle = \int_t^{t+\tau} dt' \int_t^{t+\tau} dt'' \langle \hat{d} (t') \hat{d} (t'') \hat{d} (t') \hat{d} (t') \rangle = \ldots $$

$$= \eta^2 \langle \hat{M} (t, \tau) \left[ \hat{M} (t, \tau) - 1 \right] \rangle $$

and therefore

$$\langle m(m - 1) \rangle = \eta^2 \langle \hat{M} (t, \tau) \left[ \hat{M} (t, \tau) - 1 \right] \rangle$$  

(7.6)

We can calculate now the variance photocount:

$$\langle \Delta m \rangle^2 = \eta^2 \langle \Delta \hat{M} (t, \tau) \rangle^2 + \eta (1-\eta) \langle \hat{M} (t, \tau) \rangle$$  

(7.7)

The first term of the variance photocount expresses the variance of the integrated photon number of the light beam. The second term expresses the partition noise, which is based on a random selection of a fraction $\eta$ of the incident photons by the imperfect photo-detector.

**Note:** with direct detection we are only able to retrieve properties pertaining to the incident intensity. No phase information can be retrieved.
7.2 Homodyne Detection

Homodyne detection is a method for measuring the electric field operator (actually its quadratic operator) and therefore allows phase measurement (particularly important for squeezed light because its nonclassical properties are encoded in the phase).

We have a 50\% BS, i.e. |R| = |T| = \( \frac{1}{\sqrt{2}} \) and \( \phi_R - \phi_T = \frac{\pi}{2} \). The signal that we want to measure is in arm 1 of the BS. If we place photodetectors in both output arms, then we speak of balanced homodyne detection.

We measure the difference of the two output signals, during the integration time. This corresponds to:

\[
\hat{M}_- (t, \tau) = \int_t^{t+\tau} dt' \left[ \hat{a}_3^\dagger (t') \hat{a}_3 (t') - \hat{a}_4^\dagger (t') \hat{a}_4 (t') \right] \quad (7.8)
\]

in terms of the output signal this means that

\[
\begin{pmatrix}
\hat{a}_4 \\
\hat{a}_3
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -i \\
-i & 1
\end{pmatrix}
\begin{pmatrix}
\hat{a} \\
\hat{a}_L
\end{pmatrix} \quad (7.9)
\]

where:

\[
\hat{a}_3^\dagger (t') \hat{a}_3 (t') = \frac{1}{2} \left\{ \left[ \hat{a}_3^\dagger (t') + \hat{a}_L^\dagger (t') \right] \left[ -i \hat{a} (t') + \hat{a}_L (t') \right] \right\}
\]

\[
= \frac{1}{2} \left\{ \hat{a}_3^\dagger (t') \hat{a} (t') + i \hat{a}_3^\dagger (t') \hat{a}_L (t') - i \hat{a}_L^\dagger (t') \hat{a} (t') + \hat{a}_L^\dagger (t') \hat{a}_L (t') \right\}
\]

Figure 7.2: Sketch of a balanced homodyne detector
Quantum Theory of Photodetection

\[
\hat{a}_4(t') \hat{a}_4(t') = \frac{1}{2} \{ \left[ \hat{a}^\dagger (t') + i \hat{a}_L^\dagger (t') \right] [\hat{a} (t') - i \hat{a}_L (t')] \}
\]

\[
= \frac{1}{2} \{ \hat{a}^\dagger (t') \hat{a} (t') - i \hat{a}^\dagger (t') \hat{a}_L (t') + i \hat{a}_L^\dagger (t') \hat{a} (t') + \hat{a}_L^\dagger (t') \hat{a}_L (t') \}
\]

Therefore:

\[
\hat{M}_- (t, \tau) = i \int_{t}^{t+\tau} dt' \left[ \hat{a}^\dagger (t') \hat{a}_L (t') - \hat{a}_L^\dagger (t') \hat{a} (t') \right] \tag{7.10}
\]

Although the detector measures photocount, the presence of the local oscillator makes it possible to actually measure the field!

**Note:** If we use an inefficient detector:

\[
\hat{M}_H (t, \tau) = \int_{t}^{t+\tau} dt' \left[ \hat{d}_3^\dagger (t') \hat{d}_3 (t') - \hat{d}_4^\dagger (t') \hat{d}_4 (t') \right] . \tag{7.11}
\]

Therefore the mean homodyne difference photocount is given as

\[
\langle m_- \rangle = \langle \hat{M}_H (t, \tau) \rangle = i \eta \int_{t}^{t+\tau} dt' \left\langle \hat{a}^\dagger (t') \hat{a}_L (t') - \hat{a}_L^\dagger (t') \hat{a} (t') \right\rangle \tag{7.12}
\]

and the variance is given as

\[
\langle \Delta m_- \rangle^2 = \eta^2 \Delta \langle \hat{M}_H (t, \tau)^2 \rangle + \eta (1 - \eta) \int_{t}^{t+\tau} dt' \left\langle \hat{a}_L^\dagger (t') \hat{a}_L (t') + \hat{a}^\dagger (t') \hat{a} (t') \right\rangle . \tag{7.13}
\]

Let us take a specific example:

Local oscillator: Single-mode coherent state:

\[
\alpha_L (t) = \sqrt{F_L} e^{-i \omega_L t + i \vartheta_L} \tag{7.14}
\]

where \( \omega_L \) is chosen in such a way, that \( \omega_{Signal} = \omega_L \) (homodyne). Since \( \hat{a} (t) | \alpha \rangle = \alpha (t) | \alpha \rangle \) we obtain:

\[
\langle m_- \rangle = i \eta \sqrt{F_L} \int_{t}^{t+\tau} dt' \left\langle \hat{a}^\dagger (t) e^{-i \omega_L t + i \vartheta_L} - \hat{a} (t) e^{i \omega_L t - i \vartheta_L} \right\rangle . \tag{7.15}
\]

Let us define:

\[
\hat{E}_H (\chi, t, \tau) = \frac{1}{2 \sqrt{\tau}} \int_{t}^{t+\tau} dt' \left[ \hat{a}^\dagger (t) e^{-i \omega_L t + i \vartheta_L} - \hat{a} (t) e^{i \omega_L t - i \vartheta_L} \right] . \tag{7.16}
\]
the homodyne electric field operator, where $\chi = \vartheta_L + \frac{\pi}{2}$, so that:

$$\langle m_- \rangle = 2\eta \sqrt{F_L T} \left( \hat{E}_H (\chi, t, \tau) \right)$$  \hspace{1cm} (7.17a)

$$\langle \Delta m_- \rangle^2 = \eta F_L T \left\{ 4\eta \langle \Delta \hat{E}_H (\chi, t, \tau)^2 \rangle + 1 - \eta \right\}$$  \hspace{1cm} (7.17b)

And since substantially $\hat{E} \sim \hat{x}_1$ we can conclude that with the homodyne detection we are able to measure the phase space of the field and therefore its phase properties.

**Example 1**

**Input:** single mode coherent state $\alpha (t) = \sqrt{F} e^{-i\omega t + i\vartheta_L}$ and in this case:

$$S \equiv \langle \hat{E}_H (\chi, t, \tau) \rangle = \sqrt{FT} \cos (\chi - \vartheta)$$ \hspace{1cm} (7.18)

and

$$N \equiv \langle \Delta \hat{E}_H (\chi, t, \tau)^2 \rangle = \frac{1}{4}$$ \hspace{1cm} (7.19)

With this we obtain the signal to noise ratio:

$$SNR = \frac{S^2}{N} = \frac{\langle \hat{E}_H^2 \rangle}{\langle \Delta \hat{E}_H^2 \rangle} = 4FT \cos^2 (\chi - \vartheta)$$ \hspace{1cm} (7.20)

For an inefficient detector: $SNR_H = \eta SNR$

**Example 2**

**Input:** Squeezed state

With $\hat{E}_H \sim \hat{x}_1$ follows that $\langle \hat{E}_H \rangle = 0$ for a squeezed state and therefore $\langle m_- = 0 \rangle$.

For continuous squeezed state we have that:

$$\langle \hat{a}^\dagger (\omega) \hat{a} (\omega') \rangle = \sinh^2 [s(\omega)] \delta (\omega - \omega')$$ \hspace{1cm} (7.21)

with the photon flux

$$\frac{1}{2\pi} \int d\omega \sinh^2 [s(\omega)] = F$$ \hspace{1cm} (7.22)

and

$$\langle \hat{a} (\omega) \hat{a} (\omega') \rangle = -\frac{1}{2} e^{i\vartheta (\omega)} \sinh [2s(\omega)] \delta (\omega + \omega' - \omega_p)$$ \hspace{1cm} (7.23)

**Note:** Here we used the following representation of a continuous coherent state:

$$\varphi (\omega) = s(\omega) e^{i\vartheta (\omega)}$$

therefore:
• we need to Fourier transform the operators because the properties of squeezed light are best caught in the frequency domain

• Assumption: T long enough that \( s(\omega) \) and \( \vartheta(\omega) \) vary insignificantly over \( \frac{1}{T} \)

After some calculations follows

\[
\langle \Delta \hat{E}_H (\chi, t, \tau)^2 \rangle = \frac{1}{4} \left\{ e^{2s(\omega_L)} \sin^2 \left[ \chi - \frac{1}{2} \vartheta(\omega_L) \right] + e^{-2s(\omega_L)} \cos^2 \left[ \chi - \frac{1}{2} \vartheta(\omega_L) \right] \right\}
\]

(7.24)

and we can show that

\[
0 \leq \langle \Delta \hat{E}_H (\chi, t, \tau)^2 \rangle \leq \frac{1}{4}
\]

(7.25)

and therefore

\[
0 \leq (\Delta m_-)^2 < \eta F_2 T,
\]

(7.26)

where \( F_2 \gg F \). This means that this noise derives only by the beating of the strong coherent field (of the local oscillator) with the noise in the signal.

**Note:** Shot noise of the local oscillator automatically cancels out if we use balanced homodyne detection.

# 8 Degree of Coherence and Quantum Optics

## 8.1 First Order Degree of Coherence

![Mach-Zehnder interferometer diagram](image)

Figure 8.1: Representation of a Mach-Zehnder interferometer (taken from [1])

• \( t_k = t - \frac{2k}{c} \) already contains eventual delay coming from different path lengths

• \( BS \rightarrow (r, t) \)

Intensity on output \( E_4(t) \) averaged over an oscillation cycle:

\[
\bar{T}_4 = \frac{1}{2} \varepsilon_0 c |r|^2 |t|^2 \left\{ |E_1(t_1)|^2 + |E_2(t_2)|^2 + 2 \Re [E_1^*(t_1) E_2(t_2)] \right\}
\]

(8.1)
where $E_{1,2}$ are the fields entering the second BS ($E_1 \rightarrow \text{path 1, } E_2 \rightarrow \text{path 2}$). We moreover need to average the result over the detector integration time $T$ (normally assumed to be much larger than the source’s coherence time). **DEF:**

$$\langle E(t) \rangle = \frac{1}{T} \int_0^T dt E(t)$$  \hspace{1cm} (8.2)

The intensity now becomes:

$$\langle I_4(t) \rangle = \frac{1}{2} \varepsilon_0 c |r|^2 |t|^2 \left\{ \langle |E_1(t_1)|^2 \rangle + \langle |E_2(t_2)|^2 \rangle + 2\Re \left[ \langle E_1^*(t_1) E_2(t_2) \rangle \right] \right\}$$  \hspace{1cm} (8.3)

the third term is responsible for interference and it is related to the first-order degree of coherence.

**First-Order Degree Of Coherence**

$$g^{(1)}(\tau) = \frac{\langle E^*(t) E(t+\tau) \rangle}{\langle E^*(t) E(t) \rangle}$$  \hspace{1cm} (8.4)

with $g^{(1)}(-\tau) = g^{(1)}(\tau)$ and $\tau = \frac{z_1 - z_2}{c}$. It is basically related to the fringe visibility.

By collecting $\langle I(t) \rangle = \langle E^*(t) E(t) \rangle$ we have:

$$\langle I_4(t) \rangle = \frac{1}{2} \varepsilon_0 c |r|^2 \langle I(t) \rangle \left\{ 1 + \Re \left[ g^{(1)}(\tau) \right] \right\}$$  \hspace{1cm} (8.5)

**Example: Collision Broadened Source**

We consider a source that emits radiation characterized by random phase jumps due to the collisions between the atoms of the source (see fig. 8.2).

![Phase jumps caused by collisions](image)

Figure 8.2: Phase jumps caused by collisions (taken from [1])

In this case even if each atom emits only the same frequency, the phase of the emitted field is different (assumed to be random). The total emitted electric field is
given by

$$E(t) = E_0 e^{-i\omega_0 t} \sum_k e^{-i\varphi_k(t)}$$  \hspace{1cm} (8.6)$$

Then if we calculate $g^{(1)}(\tau)$ with this definition by recalling that the correlation function that we get for the phase $\langle e^{i[\varphi_k(t+\tau) - \varphi_k(t)\rangle}$ is proportional to the probability that the atom has a period of free flight longer than $\tau$ given by:

$$p(\tau) = \frac{1}{\tau_0} \exp \left[ \frac{\tau}{\tau_0} \right]$$  \hspace{1cm} (8.7)$$

with $\tau_0 \equiv \tau_c$ being the source’s coherent time. We get at the end

$$g^{(1)}_{CB}(\tau) = \exp \left[ -i\omega_0 \tau - \frac{|\tau|}{\tau_c} \right]$$  \hspace{1cm} (8.8)$$

Figure 8.3: Degrees of first order coherence. Lorentzian shape: collision broadened source. Gaussian shape: Doppler broadened source

**Example: Doppler Broadened Source**

In this case all atoms emit coherently but the emission frequency is slightly different due to Doppler broadening, therefore

$$E(t) = E_0 \sum_k \exp [-i\omega_k t + i\varphi_k(t)]$$  \hspace{1cm} (8.9)$$
noting, that $\varphi_k$ for $k \neq j$ averages to zero (randomly distributed), after some calculations we arrive at the final result:

$$g^{(1)}_{DB}(\tau) = \exp \left[ -i\omega_0\tau - \frac{\pi}{2} \left( \frac{\tau}{\tau_c} \right)^2 \right]$$  \hspace{1cm} (8.10)

**Message:**

- $g^{(1)}(0) = 1$
- $|g^{(1)}(\tau)| = \begin{cases} 
    1 & \text{fully coherent} \\
    0 & \text{fully incoherent} \\
    \in [0,1[ & \text{partially coherent}
\end{cases}$

### 8.2 Quantum Theory of $g^{(1)}(\tau)$

**Note:** We consider single mode fields

The field operator is given by

$$\hat{E}(z,t) = \hat{E}^+(z,t) + \hat{E}^-(z,t) = \frac{i}{2} \left[ \hat{a}e^{-i(\omega t-kz)} - \hat{a}^\dagger e^{i(\omega t-kz)} \right]$$  \hspace{1cm} (8.11)

we can generalize $g^{(1)}(\tau)$ for two fields at different space-time points $(z_1,t_1)$ and $(z_2,t_2)$ write it as:

$$g^{(1)}(z_1,t_1; z_2,t_2) = \frac{\langle E^*(z_1,t_1) E(z_2,t_2) \rangle}{\sqrt{\langle |E(z_1,t_1)|^2 \rangle \langle |E(z_2,t_2)|^2 \rangle}}.$$  \hspace{1cm} (8.12)

If we want to write its quantum version, we just need to replace the fields with their operator versions.

**Intensity operator:**

$$\hat{I} \sim \hat{a}^\dagger \hat{a} \rightarrow \text{for the electric field this means } \hat{a}^\dagger \sim \hat{E}_+ \hat{a} \sim \hat{E}_- \text{ and therefore } \hat{I} \sim \hat{E}_- \hat{E}_+$$

**Operator Ordering**

we always use normal ordering ($\hat{a}^\dagger$ on the left and $\hat{a}$ on the right). The reason we do this is because with this the vacuum energy becomes zero

we consider light beams that are stationary (fluctuations of the field do not vary in time and so $g^{(1)}(\tau)$ does not depend on $\tau$)

Therefore:

$$g^{(1)}(\tau) = \frac{\langle \hat{E}^-_T(t) \hat{E}^+_T(t+\tau) \rangle}{\langle \hat{E}^-_T(t) \hat{E}^+_T(t) \rangle}$$  \hspace{1cm} (8.13)

and we can calculate the intensity as for the MZ case.
Degree of Coherence and Quantum Optics

Single Mode Field
Substitute $\hat{E}^\pm$ and calculate

$$g^{(1)}(\tau) = e^{i(\chi_1 - \chi_2)}$$  \hspace{1cm} (8.14)

with $\chi = \omega t - kz$

**Message:** We are not really able to extract new information about the nature of a quantum state by simply looking at its first order degree of coherence. Therefore any single mode field is first order coherent for all pairs of space-time points.

### 8.3 Second Order Degree of Coherence

Instead of measuring correlations in the field we now measure intensity correlations. For simplicity we assume simple polarization and that the intensity readings are taken at a fixed pint in space and at a fixed pint in space and at a fixed delay $\tau$.

$$g^{(2)}(\tau) = \frac{\langle \bar{I}(t) \bar{I}(t+\tau) \rangle}{\bar{I}^2} = \frac{\langle E^*(t) E^*(t+\tau) E(t) E(t+\tau) \rangle}{\langle E^*(t) E(t) \rangle^2}$$  \hspace{1cm} (8.15)

**Properties:**

- $g^{(2)}(-\tau) = g^{(2)}(\tau)$
- $g^{(2)}(0) \geq 1$

**Proof:** Let us consider two intensity measurements one at $t_1$ and the other one at $t_2$, so that the Cauchy-Schwarz inequality is fulfilled by

$$2\bar{I}(t_1)\bar{I}(t_2) \leq \bar{I}(t_1)^2 + \bar{I}(t_2)^2.$$  \hspace{1cm} (8.16)

**Generalization:**

$$\left\{ \frac{1}{N} \sum_k I(t_k) \right\}^2 \leq \frac{1}{N} \sum_k I(t_k)^2$$  \hspace{1cm} (8.17)

this means that for the statistical average we obtain:

$$\bar{I}^2 = \langle \bar{I}(t) \rangle^2 \leq \langle \bar{I}(t)^2 \rangle$$  \hspace{1cm} (8.18)

and therefore with

$$g^{(2)}(0) = \frac{\langle \bar{I}(t) \bar{I}(t) \rangle}{\bar{I}^2} \geq \frac{\langle I^2 \rangle}{\bar{I}^2} = 1$$  \hspace{1cm} (8.19)

we obtain $g^{(2)}(0) \geq 1$. q.e.d.

In particular it can be demonstrated that there is no upper band

$$1 \leq g^{(2)}(0) \leq \infty$$  \hspace{1cm} (8.20)
with similar arrangements we can also prove that

\[ 0 \leq g^{(2)}(\tau) \leq \infty, \tau \neq 0 \quad (8.21a) \]

and

\[ g^{(2)}(\tau) \leq g^{(2)}(0) \quad (8.21b) \]

Figure 8.4: Degree of second order coherence for various light sources. Dashed line: coherent source. Lorentzian shape: collision broadened. Gaussian shape: doppler broadened (taken from [1])

How to measure \( g^{(2)}(\tau) \)? → Hanbury-Brown-Twiss Interferometer

Figure 8.5: Schematic illustration of the Hanbury-Brown-Twiss experimental setting (taken from [2])
8.4 Quantum Theory of $g^{(2)} (\tau)$

Analogously to the case of $g^{(1)} (\tau)$ we can easily quantize the expression of $g^{(2)} (\tau)$ given above:

$$g^{(2)} (\tau) = \frac{\langle \hat{E}_T^- (t) \hat{E}_T^- (t + \tau) \hat{E}_T^+ (t + \tau) \hat{E}_T^+ (t) \rangle}{\langle \hat{E}_T^- (t) \hat{E}_T^+ (t) \rangle^2}. \quad (8.22)$$

For a single mode field, and for the case of zero delay, i.e., $\tau = 0$, this reduces to the following expression

$$g_{SM}^{(2)} (0) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}, \quad (8.23)$$

which we call single-mode second order coherence function. It is now clear that $g^{(2)} (\tau)$ will have different outcomes depending on the quantum state considered because $\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle$ gives different results for different quantum states. Let us rewrite $g^{(2)} (0)$ in a more easy way as a function of $\langle \hat{n} \rangle$ and $(\Delta n)^2$ as follows:

- $\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{n} \rangle \equiv \langle n \rangle$
- $\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}^\dagger (\hat{a} \hat{a}^\dagger - 1) \hat{a} \rangle = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \rangle = \langle n (n + 1) \rangle$

Therefore:

$$g^{(2)} (0) = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} = \frac{\langle n^2 \rangle - \langle n \rangle^2 + \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle^2} = 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2}$$

and so we obtain

$$g^{(2)} (0) = 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2}. \quad (8.24)$$

g^{(2)} (0) is therefore able to distinguish different quantum states!

8.4.1 Number States

- $\langle n \rangle = n$
- $(\Delta n)^2 = 0$

it follows

$$g_N^{(2)} (\tau) = 1 - \frac{1}{n}, \quad (8.25)$$

where $n \geq 1$. Note:

- Here $g_N^{(2)} (0) = 1 - \frac{1}{n}$, which is in contrast to the classical case, where $g^{(2)} (0) \geq 1$!
- if $|n \rangle \equiv |0 \rangle$ then $g^{(2)} (0)$ is undefined.
8.4.2 Coherent States

- \( \langle n \rangle = |\alpha|^2 \)
- \( (\Delta n)^2 = |\alpha|^2 \)

and so:

\[
g^{(2)}_{\text{C}}(0) = 1 \tag{8.26}
\]

this gives a classical result (coherent states are the most classical states among the quantum states...).

8.4.3 Chaotic Light

We did not treat chaotic light during this course. Anyway it can be characterized by a photon mean number \( \langle n \rangle \) and a variance \( (\Delta n)^2 = \langle n \rangle^2 + \langle n^2 \rangle \) and a distribution property that is essentially the Bose-Einstein statistics

\[
p_n = \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{1+n}} \tag{8.27}
\]

So we obtain

\[
|\psi\rangle = \sum p_n |n\rangle \langle n|.
\]

With this:

\[
g^{(2)}(0) = 2 \tag{8.29}
\]

it is also in accordance with the classical result (cfr. both collision/Doppler broadened).

8.4.4 Squeezed Vacuum

- \( \langle n \rangle = \sinh^2(s) \)
- \( (\Delta n)^2 = 2 \langle n \rangle (\langle n \rangle + 1) \)

so that

\[
g^{(2)}(0) = 3 + \frac{1}{\langle n \rangle} \tag{8.30}
\]

Note: When \( \langle n \rangle = 0 \) then \( g^{(2)}(0) \neq 1 \), so it does not give the coherent state limit. Care must be taken in how the vacuum limit is approached.

8.5 Comparison and Nonclassical Light

\( g^{(2)}(\tau) \) Classical:

- \( 1 \leq g^{(2)}(0) \leq \infty, \ \tau = 0 \)
- \( 0 \leq g^{(2)}(\tau) \leq \infty, \ \tau \neq 0 \)
\[ g^{(2)}(\tau) \text{ Single Mode:} \]

\[
1 - \frac{1}{\langle n \rangle} \leq g^{(2)}(\tau) \leq \infty \]

for \( \langle n \rangle \geq 1 \) and \( \forall \tau \in \mathbb{R} \)

If we compare these ranges, we can see that for the case of quantum (single mode) light we can access also the ... below ..., which is not possible in the classical case, where \( g^{(2)}(0) \geq 1 \), i.e.

\[
1 - \frac{1}{\langle n \rangle} \leq g^{(2)}(0) \leq 1 \quad (8.31) \]

for \( \langle n \rangle \geq 1 \). \textbf{Note:} If light lies in this range, then it is called nonclassical light.

---

**Figure 8.6:** Comparison second order coherence of different light forms (taken from [1])

**Why are the classical and quantum \( g^{(2)}(\tau) \) different?** Consider measuring intensity/photon number twice in succession:

\[
\langle I^2 \rangle \quad \quad \langle n(n-1) \rangle
\]

this leaves the value of I unchanged for the second measurement and therefore \( I^2 \) is to be averaged

It appears explicit the fact that there has been a first measurement followed by a second one, i.e. the measurement interferes with the measured system. While the first measurement counts \( n \) photons, the second one (affected by the first one) counts only \( (n-1) \) photons.
9 Elements of Quantum Nonlinear Optics

In the first part of this chapter we will introduce the nonlinear susceptibility, which is responsible for nonlinear processes. Furthermore the quantum optical expressions for the electric and magnetic field operators in the presence of a medium will be introduced. We will start with the classic formulation of the Maxwell equations in presence of a medium and derive the wave equation. At this point we will introduce the nonlinear polarization of the medium which is caused by the nonlinear susceptibility. In the second part we will treat nonlinear effects like second harmonic generation (SHG) and parametric up/down conversion.

9.1 Nonlinear Wave Equation

In order to derive the wave equation we need to start with the Maxwell equations in a medium:

\[
\nabla \times \mathbf{E}(r, t) = -\frac{\partial \mathbf{B}(r, t)}{\partial t}, \tag{9.1a}
\]

\[
\nabla \cdot \mathbf{D}(r, t) = \rho_{\text{free}}, \tag{9.1b}
\]

\[
\nabla \cdot \mathbf{B}(r, t) = 0, \tag{9.1c}
\]

\[
\nabla \times \mathbf{H}(r, t) = \frac{1}{c^2} \frac{\partial \mathbf{D}(r, t)}{\partial t} + \mathbf{j}. \tag{9.1d}
\]

We assume to consider a material for which the free charges \( \rho_{\text{free}} = 0 \) and the current \( \mathbf{j} = 0 \). Moreover, let us assume to consider a non-magnetic material, i.e. \( \mathbf{B} = \mu_0 \mathbf{H} \). In this case we also have that (see e.g. [3])

\[
\mathbf{D}(r, t) = \varepsilon_0 \mathbf{E}(r, t) + \mathbf{P}(r, t), \tag{9.2}
\]

where \( \mathbf{P}(r, t) \) is the polarization vector, which accounts for the effect of the material on the propagation of the electromagnetic field inside the medium.

In the following we will drop the dependence of the fields, but we will keep in mind that all fields depend on \( r \) and \( t \) if not mentioned explicitly. We obtain the wave equation by applying the curl operator two times resulting in:

\[
\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{9.3}
\]

In general the first term on the r.h.s. of this equation, i.e. \( \nabla (\nabla \cdot \mathbf{E}) \), is nonzero, because in a material \( \nabla \cdot \mathbf{D} = 0 \) but \( \nabla \cdot \mathbf{E} \neq 0 \). However, for practical cases in nonlinear optics we can treat this term as being very small, i.e. \( \nabla \cdot \mathbf{E} \approx 0 \). Furthermore with Eq. (9.2) we obtain:

\[
\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{9.4}
\]
Substituting this result into Eq. (9.3) and keeping in mind that $\nabla \cdot \mathbf{E} \approx 0$ leads to
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}.
\]

For the case of a nonlinear medium the electric polarization $\mathbf{P}(z,t)$ consists of two contributions
\[
\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL},
\]
where $\mathbf{P}_L = \varepsilon_0 \chi(\omega) \mathbf{E}$ is the linear part of the polarization, which is responsible for the refractive index. $P_{NL} = \sum_k \chi^{(k)}(\omega) E^k$ is the nonlinear part of the polarization, which is related to anharmonic terms in the Lorentz model (see [11]), e.g.
\[
\chi^{(2)}(\omega) \rightarrow P_{NL} = \chi^{(2)} \mathbf{E} \cdot \mathbf{E}.
\]

As it can be seen, the second order nonlinear polarization is proportional to the product of the square of the field, with the proportionality constant being the second order nonlinear susceptibility $\chi^{(2)}(\omega)$.

For third order processes, the nonlinear polarization can be analogously written as follows
\[
\chi^{(3)}(\omega) \rightarrow P_{NL} = \chi^{(3)} \mathbf{E} \cdot \mathbf{E} \cdot \mathbf{E},
\]
where $\chi^{(3)}(\omega)$ is the third order nonlinear susceptibility.

Substituting these expressions in Eq. (9.5) leads to
\[
\left( \nabla^2 - \frac{n(\omega)^2}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2},
\]
where we have defined the refractive index
\[
n(\omega)^2 = 1 + \chi(\omega). \quad (9.8)
\]

Normally the electric susceptibility $\chi(\omega) \in \mathbb{C}$, and therefore
\[
n(\omega) = n_1(\omega) + i n_2(\omega), \quad (9.9)
\]
where $n_1(\omega)$ describes dispersion and $n_2(\omega)$ describes absorption. For simplicity we assume $n_2(\omega) = 0$. This is a good approximation as long as we are considering a frequency range where the considered material appear transparent or quasi-transparent.

To consider a simple enough system, let us assume an electromagnetic field which propagates along the z-axis, so that $\nabla^2 \rightarrow \frac{\partial^2}{\partial z^2}$. Moreover, let us take the positive
frequency part of the Fourier expansion of the field \( E(z,t) \) in plane waves, i.e.

\[
E(z,t) = \int_0^\infty d\omega \tilde{E}(z,\omega) e^{i[\omega t - k(\omega)z]},
\]

(9.10)

with \( k(\omega) = k_0 n(\omega) \). Analogously for the nonlinear polarization we obtain

\[
P_{NL}(z,t) = \int_0^\infty d\omega \tilde{P}_{NL}(z,\omega) e^{-i\omega t}.
\]

(9.11)

Here we took the positive frequency part instead of the full Fourier transform because this would provide an easier way to write the quantum counterparts of these fields, looking back at the electric field operator, which only contains the positive frequency part, i.e. \( \hat{E}^+(z,t) \).

Applying \( \frac{\partial^2}{\partial z^2} \) on \( E(z,t) \) gives

\[
\frac{\partial^2 E}{\partial z^2} = \frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2} + 2ik(\omega) \frac{\partial \tilde{E}(z,\omega)}{\partial z} - k^2(\omega) \tilde{E}(z,\omega).
\]

(9.12)

where we used the slow varying envelope approximation (SVEA), namely we assumed

\[
\frac{\partial^2 E}{\partial z^2} \ll \frac{\partial E}{\partial z}.
\]

(9.13)

Substituting (9.11) and (9.12) in (9.5) leads to the expression

\[
2ik(\omega) \frac{\partial \tilde{E}}{\partial z} - k^2(\omega) \tilde{E} - \frac{n(\omega)^2 \omega^2}{c^2} \tilde{E} = -\mu_0 \omega^2 \hat{P}_{NL} e^{-i k(\omega)z}
\]

(9.14)

where

\[
k(\omega)^2 = k_0^2 n(\omega)^2 = \frac{\omega^2 n(\omega)^2}{c^2}
\]

(9.15)

was used. This can be written in the form

\[
\frac{\partial \tilde{E}}{\partial z} = \frac{i \omega}{2\varepsilon_0 n(\omega)} \hat{P}_{NL} e^{-i k(\omega)z},
\]

(9.16)

where we used that

\[
-\frac{\mu_0 \omega^2}{2ik(\omega)} = i \frac{\mu_0 \omega^2}{2\omega n} \cdot c = i \frac{\mu_0 \omega}{2\mu_0 \varepsilon_0 n} = i \frac{\omega}{2\varepsilon_0 n}.
\]

9.2 Quantization of the Electromagnetic Field in Dielectric Media

Since Eq. (9.16) is valid in a nonlinear medium, we need to find the correct quantization rule for the electromagnetic field in a dielectric medium in order to solve
in. In the beginning of this course we have already treated the case of quantizing the e.m. field in free space. Furthermore, we already found an expression for the quantized e.m. field for the case of the presence of an atom. Here we have assumed that the atomic medium is sufficiently diluted so that the quantized field can retain its free-space form.

For the case of the propagation of the electromagnetic field in a dielectric medium, it is not possible anymore to make the approximation of dilute gas (as in the case of a single atom interacting with the field), as the atoms in a dielectric medium are organized in a compacted crystalline structure.

To find an easy way to approach the problem of field quantization in a dielectric, we therefore need to make some assumptions, in order to be able to use the same quantization method we used for the case of free space. First of all, assume that the optical nonlinearities of the medium do not significantly affect the quantization procedure, and thus only the linear effects are included in the determination of the field operators. Furthermore we assume no absorption, i.e. \( n_2 (\omega) = 0 \). This restricts the analysis to the range of frequencies of which the medium appears transparent. The definitions of the field operator are then only valid in this frequency interval. The last assumption that we make, is that the nonlinear effects are not significant affected by the beam configurations. This means we can employ 1D-continuous mode field operators. Therefore we will consider propagation in z-direction and the polarization of the e.m. field is such that \( \hat{E} \parallel x \) and \( \hat{B} \parallel y \). The quantization procedure is essentially similar to the free-space quantization, i.e. first we calculate the field Hamiltonian \( H \) and express it as a sum of harmonic oscillators. We obtain from this calculations the following electric field operator:

\[
\hat{E}^+ (z, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \hat{E}^+ (\omega) \exp \{-i [\omega t - k (\omega) z]\},
\] (9.17)

where \( \hat{E}^+ (\omega) \) is the Fourier transformed electric field operator and \( k (\omega) \) is the k-vector in the medium, which accounts for medium dispersion through \( n (\omega) \). This result can be written in terms of \( \hat{a} (\omega) \) as follows:

\[
\hat{E}^+ (z, t) = i \int_0^\infty d\omega f (\omega) \hat{a} (\omega) \exp \{-i [\omega t - k (\omega) z]\},
\] (9.18)

where the normalization factor \( f (\omega) \) is to be determined.

The displacement operator \( \hat{D}^+ (z, t) \) in a dispersive material is obtained by means of its classical definition in a dispersive (linear) medium. The classical displacement operator is given by

\[
D (z, t) = \varepsilon_0 \varepsilon R E (z, t) = \varepsilon_0 n^2 E (z, t),
\] (9.19)
from which we obtain the nonlinear form as follows
\[
\hat{D}^+ (z,t) = i \varepsilon_0 \int_0^\infty d\omega f (\omega) n (\omega)^2 \hat{a} (\omega) \exp \left\{ -i [\omega t - k (\omega) z] \right\}.
\] (9.20)

From Maxwell’s equations we obtain the magnetic field operator as follows
\[
\hat{B}^+ (z,t) = i \int_0^\infty d\omega f (\omega) \frac{n (\omega)}{c} \hat{a} (\omega) \exp \left\{ -i [\omega t - k (\omega) z] \right\}.
\] (9.21)

To determine \( f (\omega) \) we make use of the quantum counterpart of the continuity equation for the electromagnetic field. In the classical case the energy-continuity equation reads
\[
\frac{\partial}{\partial t} \langle W (z,t) \rangle = \frac{\partial}{\partial z} \langle S (z,t) \rangle
\] (9.22)
where \( \langle W (z,t) \rangle \) is the time averaged energy density and \( \langle S (z,t) \rangle \) the time averaged Poynting vector (see [3]). From which we obtain the similar quantum optical expression of the continuity equation given by
\[
\frac{\partial \hat{W} (z,t)}{\partial t} = - \frac{\partial \hat{S}}{\partial z},
\] (9.23)
where the operators \( \hat{W} \) and \( \hat{S} \) are assumed to be normal-ordered.

**Energy Density in a Medium**

The energy density is given by
\[
W (z,t) = E \cdot D + B \cdot H = E (z,t) D (z,t) + \frac{1}{\mu_0} B (z,t) B (z,t).
\] (9.24)

Therefore we obtain the time derivative
\[
\frac{\partial W}{\partial t} = E \frac{\partial D}{\partial t} + \frac{\partial E}{\partial t} D + \frac{1}{\mu_0} B \frac{\partial B}{\partial t} + \frac{1}{\mu_0} \frac{\partial B}{\partial t} B.
\] (9.25)

Hence, we can write the time derivative of the energy operator as follows
\[
\frac{\partial \hat{W} (z,t)}{\partial t} = \hat{E}^- (z,t) \frac{\partial \hat{D}^+ (z,t)}{\partial t} + \hat{E}^+ (z,t) \frac{\partial \hat{D}^- (z,t)}{\partial t}
\]
\[
+ \frac{1}{\mu_0} \left[ \hat{B}^- (z,t) \frac{\partial \hat{B}^+ (z,t)}{\partial t} + \hat{B}^+ (z,t) \frac{\partial \hat{B}^- (z,t)}{\partial t} \right].
\] (9.26)

In the general case, it is not possible to retrieve an expression for \( \hat{W} (z,t) \) from \( (9.26) \), because the r.h.s. is not a total time derivative and cannot be written like one. This is due to the presence of the medium. However, if we first insert the
expressions of the field operators defined above and then look for an expression for \( \hat{W}(z,t) \), then Eq. (9.26) can be integrated and leads to the following result:

\[
\hat{W}(z,t) = \varepsilon_0 c \int_0^\infty d\omega \int_0^\infty d\omega' f(\omega) f(\omega') \frac{k(\omega) - k(\omega')}{\omega - \omega'} [n(\omega) + n(\omega')] \\
\times \hat{a}^\dagger(\omega) \hat{a}(\omega') e^{i(\omega - \omega')t - |k(\omega) - k(\omega')|z},
\]

(9.27)

and we assume \( f(\omega) \in \mathbb{R} \). Now we can obtain the explicit expression of the normalization factor \( f(\omega) \) by imposing that the \( z \)-integrated energy density can be written as a collection of harmonic oscillators, i.e.,

\[
A \int_{-\infty}^\infty dz \hat{W}(z,t) = \int_0^\infty d\omega \hbar \omega \hat{a}^\dagger(\omega) \hat{a}(\omega),
\]

(9.28)

where \( A \) is the space volume. Therefore we obtain

\[
f(\omega) = \sqrt{\frac{\hbar \omega}{4\pi \varepsilon_0 c A n(\omega)}}.
\]

(9.29)

**Poynting Vector**

Analogously to the classical formulation, we define the Poynting vector operator as follows

\[
\hat{S}(z,t) = \varepsilon_0 c^2 \left\{ \hat{E}^-(z,t) \hat{B}^+(z,t) + \hat{B}^-(z,t) \hat{E}^+(z,t) \right\}.
\]

(9.30)

Substituting the expression of the field operators gives

\[
\hat{S}(z,t) = \frac{\hbar}{4\pi A} \int_0^\infty d\omega \int_0^\infty d\omega' \sqrt{\frac{\omega \omega'}{n(\omega) n(\omega')}} [n(\omega) + n(\omega')] \times \\
\times \hat{a}^\dagger(\omega) \hat{a}(\omega') e^{i(\omega - \omega')t - |k(\omega) - k(\omega')|z}.
\]

(9.31)

(9.32)

If we calculate the total energy that flows through a plane of constant \( z \) we have

\[
A \int_{-\infty}^\infty dt \hat{S}(z,t) = \int d\omega \hbar \omega \hat{a}^\dagger(\omega) \hat{a}(\omega),
\]

(9.33)

since the continuity equation must be valid. Moreover, the total energy flow over the entire length of the \( z \)-axis at a given time \( t \) is given instead by

\[
A \int_{-\infty}^\infty dz \hat{S}(z,t) = \int d\omega \hbar \omega v_a(\omega) \hat{a}^\dagger(\omega) \hat{a}(\omega),
\]

(9.34)
where \( v_a(\omega) \) denotes the group velocity of the medium. It weights each frequency component of the energy flow.

Note: For a quasi-monochromatic beam, i.e. a beam with narrow spectrum distribution centred at \( \omega_0 \), the Poynting vector is given as

\[
\hat{S}(z,t) = 2\varepsilon_0 c n(\omega_0) \hat{E}^-(z,t) \hat{E}^+(z,t) = \frac{\hbar \omega_0}{A} \hat{a}^\dagger(t_R) \hat{a}(t_R),
\]

where \( t_R = t - z/v_a(\omega_0) \) is the retarded time and \( v_a(\omega) \) is the group velocity of the field inside the dielectric medium.

### 9.3 Second Harmonic Generation

In this part of our introduction we want to have a closer look at the second harmonic generation in a \( \chi^{(2)} \) medium. Therefore consider an incident, narrow-band signal at frequency \( \omega_p \), as it is shown in Fig. 9.1. Furthermore, SHG means that two photons at the frequency \( \omega_p \) are simultaneously annihilated in order to create one photon at \( 2\omega_p \).

![Figure 9.1: Schematic sketch of the SHG in a \( \chi^{(2)} \) medium, where \( \omega_p \) is the frequency of the pump.](image)

In what follows, we employ the so-called “non-depleted pump approximation”, which implies that the input (pump) beam, initially represented by a very bright coherent state, remains almost unchanged after the interaction with the nonlinear crystal. This means, for example, that if the pump beam was originally containing \( 10^9 \) photons, the SHG interaction maybe converts 50 pump photons into SHG photons, thus leaving \( 10^9 - 50 \simeq 10^9 \) photons in the pump beam after the process. The positive frequency part of the nonlinear polarization is then given by

\[
\hat{P}_{NL}^{(+)}(z,t) = \frac{\varepsilon_0}{2\pi} \int_0^\infty \int_0^\infty d\omega' d\omega'' \chi^{(2)}(\omega' + \omega'') \hat{E}^{(+)}(\omega') \hat{E}^{(+)}(\omega'') \times \exp \left\{ -i \left[ (\omega' + \omega'') t - [k(\omega') + k(\omega'')] z \right] \right\},
\]
where $\omega'$ and $\omega''$ are the frequencies of the incident light. This nonlinear polarization acts as a source for the second-harmonic signal, according to

$$\frac{\partial \hat{E}(z, \omega)}{\partial z} = \frac{i\omega}{2\varepsilon_0 n(\omega)} \hat{P}_{NL}(z, \omega) e^{-i k(\omega)z}. \quad (9.38)$$

Substituting (9.18) in the expression for the nonlinear polarization leads to

$$\hat{P}_{NL}^{(+)}(z, t) = \frac{\hbar}{4\pi c A} \int_0^\infty d\omega' \int_0^\infty d\omega'' \sqrt{\frac{\omega' \omega''}{n(\omega') n(\omega'')}} \chi^{(2)}(\omega' + \omega'') \hat{a}(\omega') \hat{a}(\omega'')$$

$$\times \exp \left\{ -i \left[ (\omega' + \omega'') t - [k(\omega') + k(\omega'')] z \right] \right\}, \quad (9.39)$$

where $A$ is the space volume.

The component of the nonlinear polarization oscillating at the frequency $\omega$ is then obtained by Fourier transform of the expression above, according to

$$P_{NL}(z, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \hat{P}_{NL}(z, \omega) e^{-i\omega t}. \quad (9.40)$$

Comparing the expressions above leads to

$$\hat{P}_{NL}(z, \omega) = \frac{\hbar}{\sqrt{2\pi c A}} \int_0^\infty d\omega' \int_0^\infty d\omega'' \sqrt{\frac{\omega' \omega''}{n(\omega') n(\omega'')}} \chi^{(2)}(\omega' + \omega'') \hat{a}(\omega') \hat{a}(\omega'')$$

$$\times \exp \left\{ i \left[ k(\omega') + k(\omega'') \right] z \right\} \delta(\omega - \omega' - \omega''). \quad (9.41)$$

With this we can solve Eq.(9.16) with respect to $\hat{E}^{(+)}(z, \omega)$

$$\hat{E}^{(+)}(L, \omega) = \frac{i\omega}{2\varepsilon_0 n(\omega)} \int_0^L dz \hat{P}_{NL}^{(+)}(z, \omega) e^{-i k(\omega)z}. \quad (9.42)$$

Using the Fourier relation we can then reconstruct the time-dependent electric field operator, i.e.

$$\hat{E}^{(+)}(L, t) = \frac{i}{\sqrt{2\pi 2\varepsilon_0 c}} \int_0^\infty d\omega \int_0^L dz \frac{\omega}{n(\omega)} \hat{P}_{NL}^{(+)}(z, \omega) e^{-i[k(\omega)z]}. \quad (9.43)$$

We can now calculate the intensity of the second harmonic signal as follows:

$$\langle \hat{I}(L, t) \rangle = \frac{1}{2\varepsilon_0 c n(\omega)} \langle \hat{E}^{(-)}(L, t) \hat{E}^{(+)}(L, t) \rangle. \quad (9.44)$$
In order to get the full expression, let us first calculate \( \hat{E}^{(+)} (L, t) \)

\[
\hat{E}^{(+)} (L, t) = \frac{i}{\sqrt{2\pi} \varepsilon_0 c} \int_0^\infty d\omega \int_0^L dz \frac{\omega}{n (\omega)} \hat{P}^{(+)}_{NL} (z, \omega) e^{-i[\omega t + k(\omega)z]}
\]

\[
= -\frac{i\hbar}{4\pi \varepsilon_0 c^2 A} \int_0^L dz \int_0^\infty d\omega' \int_0^\infty d\omega'' \int_0^\infty d\omega''' \frac{\omega}{n (\omega)} e^{-i[\omega t + k(\omega)z]}
\]

\[
\times \sqrt{\frac{\omega' \omega'''}{n (\omega') n (\omega''')}} \chi^{(2)} (\omega' + \omega'') e^{i[k(\omega') + k(\omega'')]z} \hat{a} (\omega') \hat{a} (\omega'')
\]

\[
\times \delta (\omega - \omega' - \omega''')
\]

\[
= -\frac{i\hbar}{4\pi \varepsilon_0 c^2 A} \int_0^L dz \int_0^\infty d\omega' \int_0^\infty d\omega'' \frac{\omega' + \omega''}{n (\omega' + \omega'')} e^{-i[\omega' + \omega'' + k(\omega') + k(\omega'')]z}
\]

\[
\times \sqrt{\frac{\omega' \omega'''}{n (\omega') n (\omega''')}} \chi^{(2)} (\omega' + \omega''') e^{i[k(\omega') + k(\omega'')]z} \hat{a} (\omega') \hat{a} (\omega'')
\]

If we assume that the pump beam can be considered having a narrow-band spectrum centered around the frequency \( \omega_p \) (quasi-monochromatic approximation), then we can set \( \omega' \approx \omega_p \) and \( \omega''' \approx \omega_p \) in the quantities appearing inside the integral above, except for the operators and time-dependent exponential. By doing so we obtain the following result:

\[
\hat{E}^{(+)} (L, t) = \left( -\frac{i\hbar}{8\pi \varepsilon_0 c^2 A} \right) \int_0^L dz \frac{2\omega_p}{n (2\omega_p)} e^{-i[k(2\omega_p + 2k(\omega_p))z]}
\]

\[
\times \frac{\omega_p}{n (\omega_p)} \chi^{(2)} (2\omega_p) \int_0^\infty d\omega' \hat{a} (\omega') e^{-i\omega' t} \int_0^\infty d\omega'' \hat{a} (\omega'') e^{-i\omega'' t}.
\]  \( (9.45) \)

By noting that the integrals in \( \omega' \) and \( \omega''' \) are essentially the Fourier transform of the time dependent operators \( \hat{a} (t) \), we can then rewrite the previous equation as follows

\[
\hat{E}^{(+)} (L, t) = \left( -\frac{i\hbar \omega_p^2}{2\pi \varepsilon_0 c^2 A} \right) \frac{1}{n (2\omega_p) n (\omega_p)} \chi^{(2)} (2\omega_p) \hat{a} (t) \hat{a} (t) \int_0^L dz e^{-i[k(2\omega_p + 2k(\omega_p))z]}
\]

\( (9.46) \)

We obtain an equivalent result for \( \hat{E}^{(+)} (L, t) \), which substituted in the equation for
the intensity of the second harmonic reads as follows:

\[
\langle \hat{I}_{\text{SHG}} (L, t) \rangle = 2 \varepsilon_0 cn (2 \omega_p) \left( \hat{E}^- (L, t) \hat{E}^+ (L, t) \right) \\
= 2 \varepsilon_0 cn (2 \omega_p) \left( -\frac{i \hbar \omega_p^2}{4 \pi \varepsilon_0 c^2 A} \right) \left( \frac{i \hbar \omega_p^2}{4 \pi \varepsilon_0 c^2 A} \right) \\
\times |\chi^{(2)}(2 \omega_p)|^2 \left[ \frac{1}{n(2 \omega_p) n(\omega_p)} \right]^2 \\
\times \left| \int_0^L \! dz e^{-i[k(2 \omega_p) + 2k(\omega_p)]z} \right|^2 \\
\times \langle \hat{a}^\dagger (t) \hat{a}^\dagger (t) \hat{a} (t) \hat{a} (t) \rangle. \quad (9.47)
\]

Defining $\Delta k = k(2 \omega_p) - 2k(\omega_p)$ we have

\[
\left| \int_0^L \! dz e^{-i \Delta k z} \right|^2 = \int_0^L \! dz e^{i \Delta k z} \int_0^L \! d\zeta e^{-i \Delta k \zeta} \\
= \left( \frac{i}{\Delta k} \right) \left( \frac{-i}{\Delta k} \right) [e^{-i \Delta k z}]_0^L [e^{-i \Delta k \zeta}]_0^L \\
= \frac{1}{\Delta k^2} [e^{-i \Delta k L} - 1] [e^{i \Delta k L} - 1] \\
= \left( \frac{1}{\Delta k} \right)^2 \{1 - e^{-i \Delta k L} - e^{i \Delta k L} + 1\} \\
= \frac{2}{\Delta k^2} \{1 - \cos (\Delta k L)\} \\
= \frac{2}{\Delta k^2} \left[ 1 - 1 + 2 \sin^2 \left( \frac{\Delta k L}{2} \right) \right] \\
= \frac{\sin^2 (\Delta k L/2)}{(\Delta k L/2)^2}.
\]

Summarizing everything leads to the expression for the time averaged second harmonic intensity

\[
\langle \hat{I}_{\text{SHG}} (L, t) \rangle = \frac{\hbar^2 \omega_p^4}{8 \pi^2 \varepsilon_0 c^2 A^2} \left[ \frac{|\chi^{(2)}(2 \omega_p)|^2}{n(2 \omega_p) n(\omega_p)^2} \right] \sin^2 (\Delta k L/2) \\
\times \chi^{(2)}(2 \omega_p) \langle \hat{a}^\dagger (t) \hat{a}^\dagger (t) \hat{a} (t) \hat{a} (t) \rangle. \quad (9.48)
\]

If we consider a continuous wave at the frequency $\omega_p$ we can write $\langle \hat{a}^\dagger (t) \hat{a}^\dagger (t) \hat{a} (t) \hat{a} (t) \rangle$ as a second order correlation function $g^{(2)}_{p,p} (0)$, i.e.

\[
\langle \hat{a}^\dagger (t) \hat{a}^\dagger (t) \hat{a} (t) \hat{a} (t) \rangle = g^{(2)}_{p,p} (0) \langle \hat{a}^\dagger (t) \hat{a} (t) \rangle^2. \quad (9.49)
\]

The second harmonic signal is proportional to $g^{(2)} (0)$ and to the square of the mean photon flux of the incident pump beam. This is the case, because, as already mentioned SHG is the process of simultaneous two photon absorption at the frequency...
\( \omega_p \). Furthermore, \( g^{(2)}(\tau) \) is a measure for two photon absorption. Therefore, if \( g^{(2)}(\tau) \) of the pump is zero, we will not obtain second harmonic generation, e.g. consider a single photon state \( |1\rangle \) as pump. There is no second photon to absorb and therefore \( I_{SHG} = 0 \) and also \( g^{(2)}(\tau) = 0 \) for \( |1\rangle \), since \( g^{(2)}(0) = 1 - 1/n \) for Fock states.

Moreover, for a thermal source we obtain

\[
g^{(2)}_{THERMAL}(0) = 2 \neq g^{(2)}_{COH}(0),
\]

since the availability of photon pairs in thermal light is greater than in coherent light. This leads to the fact, that \( I_{SHG}^{(thermal)}/I_{SHG}^{(coh)} = 2 \).

In anti-bunched light (e.g. Fock states), we have that:

\[
g^{(2)}_{A.B.}(0) < g^{(2)}_{COH}(0),
\]

therefore, anti-bunched light will generate less SHG than coherent light.

**Phase Matching**

For phase matching we need to consider the \( L \)-dependent term of the intensity, i.e.

\[
\frac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}.
\]

If \( \Delta k = 0 \) this term becomes one, and the second harmonic intensity \( I_{SHG} \) reaches its maximum. This means that for \( \Delta k = 0 \) we obtain

\[
k(2\omega_p) = 2k(\omega_p),
\]

which is the phase matching condition. This corresponds to the conservation of momentum in passing from \( \omega_p \) to \( 2\omega_p \).
9.4 Parametric Down Conversion

Figure 9.2: Schematic sketch of the quantum optical parametric down conversion in a $\chi^{(2)}$ medium, where $\omega_p$ is the frequency of the pump, $\omega_i$ is the idler frequency and $\omega_s$ the signal frequency.

The nonlinear polarization is given by

$$\hat{P}^{(+)}_{NL} (z, t) = \frac{\varepsilon_0}{2\pi} \int_0^\infty d\omega' \int_0^\infty d\omega'' \chi^{(2)} (\omega'' - \omega') \hat{E}^{(-)} (z, \omega') \hat{E}^{(+)} (\omega'') \times \exp \left\{ i \left[ (\omega' - \omega'') t - [k (\omega') - k (\omega'')] \right] z \right\},$$  \hspace{1cm} (9.51)

where $\omega'$ and $\omega''$ are the frequencies of the signal and the pump, respectively. If we take the Fourier transform of the nonlinear polarization and evaluate it at the idler frequency $\omega_i = \omega_p - \omega$ we have:

$$\hat{P}^{(+)}_{NL} (z, \omega_p - \omega) = \frac{\varepsilon_0}{\sqrt{2\pi}} \int_0^\infty d\omega' \int_0^\infty d\omega'' \chi^{(2)} (\omega'' - \omega') \hat{E}^{(-)} (z, \omega') \hat{E}^{(+)} (\omega'') \times e^{-i[k(\omega') - k(\omega'')]z} \delta (\omega_p - \omega + \omega' - \omega'').$$  \hspace{1cm} (9.52)

For our calculations we will assume that the pump is a bright coherent state, namely

$$\hat{a} (\omega'') | \alpha_p \rangle = \alpha_p (\omega'') | \alpha_p \rangle.$$  \hspace{1cm} (9.53)

Furthermore, we assume that we have quasi monochromatic light, i.e., we have a narrow band around $\omega_p$ which we express as

$$\alpha_p (\omega'') = \sqrt{2\pi F_p} e^{i\vartheta_p} \delta (\omega'' - \omega_p),$$  \hspace{1cm} (9.54)

where $F_p$ is the mean photon flux and $\vartheta_p$ is the pump phase. The phase matching condition is given by

$$\Delta k = k (\omega_p) - k (\omega) - k (\omega_p - \omega),$$  \hspace{1cm} (9.55)
where we have for $\Delta k \neq 0$ the non-degenerate and for $\Delta k = 0$ the degenerate parametric down conversion, respectively.

Since signal and idler have different frequencies, we define two different field operators (they are, in fact, two different field modes):

$$\hat{E}^{(+)}(z, \omega) = i \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 c A_n (\omega)}} \hat{a}_z (\omega) ,$$  \hspace{1cm} (9.56a)

as electric field operator of the signal and

$$\hat{E}^{(+)}(z, \omega_p - \omega) = i \sqrt{\frac{\hbar (\omega_p - \omega)}{2 \varepsilon_0 c A_n (\omega_p - \omega)}} \hat{b}_z (\omega) ,$$  \hspace{1cm} (9.56b)

is the electric field operator of the idler. They both obey (9.16), i.e.

$$\frac{\partial \hat{E}^{(+)} (z, \omega_p - \omega)}{\partial z} = \frac{\omega_p - \omega}{n (\omega_p - \omega)} \sqrt{\frac{\hbar \omega_p F_p}{8 \varepsilon_0 c^3 A_n (\omega_p)}} \chi^{(2)} (\omega_p - \omega) \hat{E}^{(-)} (z, \omega) \times \exp \left\{ i \left[ \vartheta_p + (k (\omega_p) - k (\omega) - k (\omega_p - \omega)) z \right] \right\} ,$$  \hspace{1cm} (9.57a)

and

$$\frac{\partial \hat{E}^{(-)} (z, \omega_p)}{\partial z} = - \frac{\omega}{n (\omega)} \sqrt{\frac{\hbar \omega_p F_p}{8 \varepsilon_0 c^3 A_n (\omega_p)}} \chi^{(2)}^* (\omega) \hat{E}^{(+)} (z, \omega_p - \omega) \times \exp \left\{ -i \left[ (k (\omega_p) - k (\omega) - k (\omega_p - \omega)) z \right] \right\} .$$  \hspace{1cm} (9.57b)

This set of coupled wave equations can be written with respect to $\hat{a}$ and $\hat{b}$. After introducing

$$s (\omega) e^{i \vartheta (\omega)} = - \sqrt{\frac{\hbar \omega_p \omega (\omega_p - \omega) F_p}{8 \varepsilon_0 c^3 A_n (\omega_p) n (\omega) n (\omega_p - \omega)}} \chi^{(2)} (\omega_p - \omega) e^{i \vartheta_p L}$$  \hspace{1cm} (9.58)

we obtain

$$\frac{\partial \hat{b}_z (\omega_p - \omega)}{\partial z} = - \frac{s (\omega)}{L} e^{i \vartheta (\omega)} \hat{a}_z^\dagger (\omega) ,$$  \hspace{1cm} (9.59a)

and

$$\frac{\partial \hat{a}_z (\omega)}{\partial z} = - \frac{s (\omega)}{L} e^{-i \vartheta (\omega)} \hat{b}_z (\omega_p - \omega) .$$  \hspace{1cm} (9.59b)

whose solution is given by

$$\hat{a}_L (\omega) = \hat{a}_0 (\omega) \cosh \left[ s (\omega) \right] - \hat{b}_0 (\omega_p - \omega) e^{-i \vartheta (\omega)} \sinh \left[ s (\omega) \right] ,$$  \hspace{1cm} (9.60a)

and

$$\hat{b}_L (\omega_p - \omega) = \hat{b}_0 (\omega_p - \omega) \cosh \left[ s (\omega) \right] - \hat{a}_0^\dagger (\omega) e^{i \vartheta (\omega)} \sinh \left[ s (\omega) \right] ,$$  \hspace{1cm} (9.60b)
for the case of non-degenerate parametric down conversion.
For the degenerate case we have the same theory as before, just with \( \hat{a} \equiv \hat{b} \). Therefore, the solution of the coupled equations becomes

\[
\hat{a}_L^\dagger (\omega) = \hat{a}_0^\dagger (\omega) \cosh [s (\omega)] - \hat{a}_0 (\omega_p - \omega) e^{i\theta(\omega)} \sinh [s (\omega)], \tag{9.61}
\]

which corresponds to a single mode squeezed state.
This means, that PDC and degenerate PDC are natural sources of squeezed light (and they possess all the non-classical properties of squeezed light). Therefore we can write

\[
|\psi\rangle \xrightarrow{PDC} |\xi (\omega)\rangle, \tag{9.62}
\]

with

\[
\xi (\omega) = s (\omega) e^{i\theta(\omega)}. \tag{9.63}
\]

10 Appendix A: The Harmonic Oscillator

The Hamiltonian of a harmonic oscillator is given by

\[
\hat{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2. \tag{10.1}
\]

To solve the problem, we introduce the so called creation (\( \hat{a}^\dagger \)) and annihilation (\( \hat{a} \)) operators, defined as follows:

\[
\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \tag{10.2a}
\]

\[
\hat{p} = i \sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a}). \tag{10.2b}
\]

Substituting this into Eq. (10.1), we obtain the so-called second quantized form of the harmonic oscillator Hamiltonian, i.e.,

\[
\hat{H} = \frac{\hbar \omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) \tag{10.3}
\]

with the commutation rules

\[
[\hat{a}, \hat{a}^\dagger] = \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1 \tag{10.4a}
\]

\[
[\hat{a}, \hat{a}] = 0 = [\hat{a}^\dagger, \hat{a}^\dagger]. \tag{10.4b}
\]

Using the commutation rule (10.4), the Hamiltonian of the harmonic oscillator can be rewritten as follows:

\[
\hat{H} = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \tag{10.5}
\]
Appendix B: The Quantization of a Paraxial Electromagnetic Field

The quantization procedure described in the first chapters of these notes has been carried for the electromagnetic field in free space, with the aid of a fictitious cavity, whose main aim was to introduce a suitable (and regular) set of modes upon which the general electromagnetic field could be expanded. In carrying out that quantization procedure, however, no attention at all was put on the spatial properties of the field itself, as the main result of the quantization of the electromagnetic field was the introduction of Fock states, i.e. energy eigenstates associated to the field. The analysis carried out throughout all these notes, therefore, only focused on the energy-eigenstates description of the quantized field, and the spatial structure of the field itself has not been taken into account, assuming that the field could always be represented by a plane wave in space.

In this last chapter, however, we intend to overcome this limitation, by discussing how we can quantify a paraxial electromagnetic field, and how can we introduce field operators, which are able to create, for example, a single photon in a given spatial mode (for example, in a Hermite-Gauss or Laguerre-Gauss mode). To do that, however, we need to reconsider the problem of field quantization, putting particular attention to the fact that the quantized field must be a solution of the paraxial equation, and therefore the correspondent field state must be defined in a suitably chosen Hilbert space, which guarantees the feasibility of the quantization procedure and at the same time ensures the field to be a solution of the paraxial equations.

The problem of paraxial quantization was first solved by Deutsch and Garrison in 1991 [12] and can be found (in a slightly more didactical version) in Chapter 6 of Ref. [13]. However, the approach presented in Ref. [14] is much more clear and easier to adapt to the scope of these notes. Therefore, the procedure of quantization of a paraxial light beam will be explained in this chapter taking Ref. [14] as a main guidance. The results presented here, however, are in complete agreement with the work by Deutsch and Garrison presented in Ref. [12].

11.1 From the Vector Potential to the Paraxial Operator

As a starting point for our analysis, let us consider the expression of the positive frequency part of the quantized vector potential in the Coulomb gauge, for a continuous
Appendix B: The Quantization of a Paraxial Electromagnetic Field

mode field, i.e.,

$$\hat{A}^{(+)}(r, t) = \sum_{\lambda=1}^{2} \int d^3k \sqrt{\frac{\hbar}{16\pi^3c\varepsilon_0|k|}} e^{i(\lambda)(k)} e^{i(kr-ct)|k|}. \quad (11.1)$$

To correctly carry out the quantization procedure for a paraxial beam, we use a trick similar to the one used for the general case in chapter 1, i.e., we introduce a longitudinal quantization length $L$, so that the longitudinal component $k_z$ of the field’s wave vector can be discretized as follows: $\zeta_n = (2\pi/L)n$, where $n \geq 0$ selects only forward propagating waves. By doing this, the $k_z$-integration in the above equation reduces to a summation over the allowed values of $n$, and only the integration over the transverse momentum $k_T = k_x \hat{x} + k_y \hat{y}$ remains. Therefore

$$\int d^3k \to \frac{2\pi}{L} \sum_n \int d^2k_T,$$

where the prefactor $2\pi/L$ has been inserted to correctly account for normalization. Together with this substitution, we must also redefine the annihilation operator appearing in Eq. (11.1) as follows

$$\hat{a}_\lambda(k) \to \sqrt{\frac{L}{2\pi}} \hat{a}_\lambda(k, n),$$

in such a way that the bosonic commutation relation

$$[\hat{a}_\lambda(k, n), \hat{a}_\mu^\dagger(k', m)] = \delta_{\lambda\mu} \delta_{nm} \delta(k - k') \quad (11.2)$$

continues to be valid. After these transformations, Eq. (11.1) can be then rewritten as follows:

$$\hat{A}^{(+)}(r, t) = \sum_{\lambda, n} \int d^2k_T \sqrt{\frac{\hbar}{8\pi^2c\varepsilon_0|k|L}} e^{i(\lambda)(k)} \hat{a}_\lambda(k_T, n)e^{i(k_0z-\omega_0t)}. \quad (11.3)$$

Our aim is to be able to write the expression of the vector potential operator in such a way that it could be interpreted as a solution of the paraxial wave equation. To do that, we should recall that a paraxial field can be always written as the product of an envelope function $\psi(x, y, z)$, which varies slowly with $z_0$ and a plane wave term $\exp \left[i(k_0z-\omega_0t)\right]$, which accounts for the propagation along the $z$ direction and the harmonic time dependence of the field itself. A closer inspection to Eq. (11.3), however, reveals that we do not have such exponential factor.

However, if we introduce the following, trivial identity

$$1 = \frac{\mathcal{L}}{2\pi} \int_0^{\mathcal{L}/2\pi} dk_0 \frac{e^{i(k_0z-\omega_0t)}}{e^{i(k_0z-\omega_0t)}},$$

we can rewrite Eq. (11.3) in a form that is consistent with the paraxial wave equation.
into Eq. \eqref{eq:11.3}, we can rewrite it in the following form:

\begin{equation}
\hat{A}(r, t) = \frac{L}{2\pi} \int_{0}^{L/2\pi} dk_{0} e^{-i(k_{0}z - \omega_{0}t)} \hat{\psi}(r, t), \tag{11.4}
\end{equation}

where

\begin{equation}
\hat{\psi}(r, t) = \sum_{\lambda, n} \int d^{2}k T \sqrt{\frac{\hbar}{8\pi^{2}c\varepsilon_{0}}} \hat{e}^{(\lambda)}(k) \hat{a}_{\lambda}(k_{T}, n) e^{i(k - k_{0}z - \omega_{0}t)} e_{\lambda}(k T, n), \tag{11.5}
\end{equation}

and \( k = k_{T} + \zeta_{n} \hat{z} \).

Written in this form, Eq. \eqref{eq:11.4} is an exact solution of the paraxial wave equation, if the field operator \( \hat{\psi}(r, t) \) is itself an exact solution of the paraxial equation, i.e.,

\begin{equation}
\left( 2ik_{0} \frac{\partial}{\partial z} + \nabla_{T}^{2} \right) \hat{\psi}(r, t) = 0. \tag{11.6}
\end{equation}

Substituting Eq. \eqref{eq:11.5} into the above equation brings to the following dispersion relation:

\begin{equation}
\frac{\zeta_{n}}{k_{0}} = 1 - \frac{|k_{T}|^{2}}{2k_{0}^{2}}, \tag{11.7}
\end{equation}

which is the usual paraxial dispersion relation \cite{16}. It is worth noticing that the above relation constitutes a constraint on the allowed valued of \( \zeta_{n} \) that a paraxial wave can take, as a function of the transverse wave vector \( k_{T} \). This, in our case, constitutes a constraint that defines a two dimensional sub-space in the \( k \)-space, where Eq. \eqref{eq:11.4} is at the same time an exact solution of the paraxial equation (because of the way we defined \( \hat{\psi}(r, t) \)) and also an exact solution of the wave equation, a condition which is necessary for being able to write the quantized form of the vector potential as given by Eq. \eqref{eq:11.1}. This 2D sub-space is the paraxial Hilbert space introduced in Ref. \cite{12}. In what follows, we will try to rewrite Eq. \eqref{eq:11.4} in a form which automatically accounts for this constraint, thus making not necessary to introduce the paraxial Hilbert space explicitly, thus making the quantization procedure easier to understand.

### 11.2 The Paraxial Dispersion Relation

To implement a form of the paraxial operator \( \hat{\psi}(r, t) \) which automatically accounts for the paraxial constraint, we first need to elaborate a bit more eq. \eqref{eq:11.7}. To do that, let us rename the transverse wave vector as \( q = k_{T} \) and let us define \( q = |k_{T}| \). Moreover, let us introduce the following quantity

\begin{equation}
\theta \sqrt{2} = \frac{q}{k_{0}}, \tag{11.8}
\end{equation}

and
which is a generalization of the well known divergence angle of a Gaussian beam [16]. By then introducing the angle $\phi$ formed by the propagation direction $\hat{z}$ and the field $k$-vector $k$, it is not difficult to prove that

$$\tan \phi = \frac{q}{k \cdot \hat{z}} = \frac{q}{\zeta_n}.$$  

Using this result and Eq. (11.8) we can rewrite the dispersion relation (11.7) in the following, dimensional form:

$$\theta \sqrt{2} = - \cot \phi + \sqrt{2 + \cot^2 \phi}. \quad (11.9)$$

The above equation is exact and links the beam divergence $\theta \sqrt{2}$ with the angle formed by the beam with the propagation axis. For the paraxial case, $\phi \ll 1$, and a Taylor expansion of the above relation brings to the following result

$$\theta \sqrt{2} \simeq \phi - \frac{\phi^3}{6} + O(\phi^5). \quad (11.10)$$

With this approximation, the dispersion relation (11.7) becomes

$$\zeta_n \simeq k_0 (1 - \theta^2). \quad (11.11)$$

We can use this result to write the $k$-vector and the polarization vectors appearing in Eq. (11.5) as $\theta$-dependent quantities, in such a way that it becomes very easy to isolate their paraxial contribution. In fact, in analogy with paraxial gaussian beams [17], the purely paraxial terms are the one independent on $\theta$, and the terms which depend from $\theta^2$ represent the lowest order corrections to the paraxial approximation.

Using Eq. (11.10) we can then write the following, approximated, quantities:

$$k \rightarrow k \simeq q \hat{q} + k_0 (1 - \theta^2) \hat{z}, \quad (11.12a)$$

$$\hat{e}^{(2)}(k) \rightarrow \hat{e}^{(2)}(q, \theta) \simeq \hat{z} \times \hat{q}, \quad (11.12b)$$

$$\hat{e}^{(1)}(k) \rightarrow \hat{e}^{(1)}(q, \theta) \simeq \hat{e}^{(1)}(q, \theta) \times k. \quad (11.12c)$$

Moreover, Eq. (11.11) needs to be compared with the longitudinal quantization condition $\zeta_n = (2\pi/L)n$ that we assumed at the beginning of our analysis. Since the allowed longitudinal $k$-vectors are only the ones which have the form $\zeta_n = (2\pi/L)n$, the paraxial dispersion relation given by Eq. (11.11) forces the integer $n$ to acquire a precise value, i.e.

$$n \equiv n(\theta) = \left[ \frac{k_0 L}{2\pi} (1 - \theta^2) \right]_{I.P.}, \quad (11.13)$$

where the subscript $I.P.$ stands for “Integer Part”.

This result is exactly what we were looking for. Among all the possible value
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of $\zeta_n$ deriving from the introduction of a longitudinal quantization length $L$, the constraint imposed by $\hat{\psi}(\mathbf{r}, t)$ being an exact solution of the paraxial equation selects only one single value of $\zeta_n$, namely the one corresponding to $n = n(\theta)$. If we now insert artificially a condition of the kind $\delta_{n,n}(\theta)$ in the expression of $\hat{\psi}(\mathbf{r}, t)$ given by Eq. (11.15), we therefore have a field operator which automatically accounts for the paraxial constraint (11.10). The final expression of the paraxial field operator $\hat{\psi}(\mathbf{r}, t)$ is then given by

$$\hat{\psi}(\mathbf{r}, t) \sum_{\lambda, n} \int d^2k_T \sqrt{\frac{\hbar}{8\pi^2c\varepsilon_0|\mathbf{k}|L}} \hat{e}^{(\lambda)}(\mathbf{k})\hat{a}_\lambda(\mathbf{k}_T, n)\delta_{n,n(\theta)}e^{i[k_0z - (c|\mathbf{k}| - \omega)t]}.$$  (11.14)

11.3 Back to Continuous Mode Operators

Now that we have obtained our desired result of writing the expression of a field operator, which is automatically defined (through the paraxial constraint $\delta_{n,n(\theta)}$) in a suitable paraxial Hilbert space, we can restore the continuous character of the longitudinal wave vector $k_z$. We have in fact discretized the longitudinal component of the wave vector only for convenience, in order to deal with well defined quantities (i.e., in order to avoid singularities and problems with infinities).

If we then make the following substitutions, where now $k_z$ has been replaced with $c\omega$:

$$\sum_n \rightarrow \frac{L}{2\pi c} \int d\omega,$$

$$\hat{a}_\lambda(\mathbf{q}, n) \rightarrow \sqrt{\frac{2\pi c}{L}} \hat{a}_\lambda(\mathbf{q}, \omega),$$  (11.15)

and we take the continuous limit of $\delta_{n,n(\theta)}$ as

$$\delta_{n,n(\theta)} \rightarrow \frac{2\pi c}{L} \delta(\omega - cz(1 - \theta^2)).$$  (11.16)

we can rewrite the expression of the vector potential operator (11.1) as follows:

$$\hat{A}^{(+)}(\mathbf{r}, t) = \sum_{\lambda} \int d\omega e^{-i\omega(t-z/c)} \sqrt{\frac{\hbar}{16\pi^3\varepsilon_0 c\omega}}$$

$$\times \int d^2q \hat{e}^{(\lambda)}(\mathbf{q}, \omega, z, t)\hat{a}_\lambda(\mathbf{q}, \omega)e^{i[q\cdot\mathbf{x} - q^2cz/(2\omega)]},$$  (11.17)

where $\mathbf{x} = x\hat{x} + y\hat{y}$ and

$$\hat{e}^{(\lambda)}(\mathbf{q}, \omega, z, t) = \hat{e}^{(\lambda)}(\mathbf{q}, \theta) \left( \frac{1 + \theta^2}{\sqrt{1 + \theta^4}} \right)^{1/2} e^{-i\omega t(\sqrt{1 + \theta^4} - 1)}.$$  (11.18)

One can easily prove that Eq. (11.17) represents a bona fide vector potential, which for $t = 0$ is an exact solution of the paraxial equation and for $t \geq 0$ is an exact
solution of the wave equation. The vector potential operator (11.17) is therefore defined in the paraxial Hilbert space defined by the paraxial constraint (11.10).

11.4 The Maxwell-Paraxial Modes

Equation (11.17) is already a good form of paraxial vector potential operator. However, the annihilation operator that appears there is written in the frequency space, as it annihilates a photon with a specific transverse momentum $q$. To be able to manage the creation and annihilation of photons in spatial modes of the field, however, we will need to rewrite Eq. (11.17) in such a way that it contains operators defined in the representation space spanned by $x$.

To do that, let us introduce the following Fourier transform for the operators $\hat{a}_\lambda(q, \omega)$:

$$\hat{a}_\lambda(x, \omega) = \frac{1}{2\pi} \int d^2q \hat{a}_\lambda(q, \omega) e^{iq \cdot x}. \quad (11.19)$$

It is left to the reader to prove that $\hat{a}_\lambda(x, \omega)$ obeys the bosonic commutation rule

$$[\hat{a}_\lambda(x, \omega), \hat{a}_\mu^\dagger(y, \Omega)] = \delta_{\lambda\mu} \delta(\omega - \Omega) \delta(x - y). \quad (11.20)$$

Substituting Eq. (11.19) into Eq. (11.17) and calling $k_0 = \omega/c$ gives the following, final, result:

$$\hat{A}^+(r, t) = \sum_\lambda \int d\omega \frac{e^{i(\omega t - z/c)}}{\sqrt{4\pi\varepsilon_0 c\omega/\hbar}} \hat{\mathcal{A}}^+(x, z, \omega, t), \quad (11.21)$$

where

$$\hat{\mathcal{A}}^+(x, z, \omega, t) = \int d^2x' F^{(\lambda)}(x', z, x, \omega, t) \hat{a}_\lambda(x', \omega), \quad (11.22)$$

with $F^{(\lambda)}(x', z, x, \omega, t)$ are the so-called Maxwell-paraxial modes, whose explicit expression is given by

$$F^{(\lambda)}(x', z, x, \omega, t) = \frac{1}{(2\pi)^2} \int d^2q e^{(\lambda)}(q, \omega, z, t)e^{i\left(q \cdot (x - x') - q^2cz/(2\omega)\right)} \quad (11.23)$$

Let us comment this result. First of all, notice that the Maxwell-paraxial modes $F^{(\lambda)}(x', z, x, \omega, t)$ represent the field at a plane $z$ at time $t$ generated by a point source located at $x'$ in the transverse plane $z = 0$, oscillating at the paraxial frequency $\omega$. From this perspective, $F^{(\lambda)}(x', z, x, \omega, t)$ can be interpreted as the quantum counterpart of the Huygens-Fresnel diffraction integral. Moreover, if the paraxial approximation ids valid, i.e, if the integration in Eq. (11.23) is limited to the domain $C_\omega = \{q : q \ll \omega/c\}$, the Maxwell-paraxial modes reduce to

$$F^{(\lambda)}(x', z, x, \omega, t) = \int_{C_\omega} \frac{d^2q}{(2\pi)^2} e^{(\lambda)}(q) e^{i\left(q \cdot (x - x') - q^2cz/(2\omega)\right)}, \quad (11.24)$$

which is nothing but the paraxial propagator, i.e. the Fresnel diffraction integral.
Moreover, the Maxwell-paraxial modes are quasi-orthogonal, in the sense that

\[
\int d^2x \ F^{(\lambda)}(x', z, x, \omega, t) F^{(\mu)}(x'', z, x, \omega, t) = \delta_{\lambda \mu}
\]

\[
\times \int \frac{d^2q}{(2\pi)^2} \left( \frac{1 + \theta^2}{\sqrt{1 + \theta^4}} \right)^{1/2} e^{iq(x' - x')}.
\]

(11.25)

Notice that these modes become fully orthogonal in the paraxial regime.

### 11.5 The Paraxial States of the Electromagnetic Field

Let us now concentrate on Eq. (11.22) and its meaning. According to Eq. (11.22), the Fourier component at frequency \( \omega \) of the paraxial vector potential operator \( \hat{A}^{(+)}(r, t) \) is given as the superposition of all the annihilation operators at the position \( x' \), weighted by the corresponding Maxwell-paraxial mode \( F^{(\lambda)}(x', z, x, \omega, t) \).

This means, from a very general point of view, that if we want to create a photon in a certain paraxial mode given by \( F^{(\lambda)}(x', z, x, \omega, t) \) (or, equivalently, its paraxial pointerpart \( P^{(\lambda)}(x', z, x, \omega, t) \)), we need to construct a field operator, which is given by the superposition of all the field operators that create a photon in the point of space \( x' \), each of them weighted by the value of the Maxwell-paraxial mode at that spatial point.

If we then introduce the following single photon paraxial state

\[
|x, \omega, \lambda\rangle = \hat{a}^\dagger_{\lambda}(x, \omega)|0\rangle,
\]

(11.26)

then

\[
\langle 0| \hat{A}^{(+)}(r, t)|x, \omega, \lambda\rangle \simeq F^{(\lambda)}(x', z, x, \omega, t)e^{-i\omega(t - z/c)},
\]

(11.27)

and therefore the Maxwell-paraxial modes can be interpreted as the single photon wave function. The single photon state defined in Eq. (11.26) can also be used to find the spatial representation of any state of the electromagnetic field. Given in fact a general state \( |\psi\rangle \) of the electromagnetic state, the projection onto \( |x, \omega, \lambda\rangle \) gives in fact

\[
\langle x, \omega, \lambda|\psi\rangle = \psi_{\lambda}(x, \omega),
\]

(11.28)

which is an exact solution of the paraxial equation.

### 11.6 An Example of Paraxial Mode Operators

As it is well known [16], the fundamental solution of the paraxial equation is given in the form of a Gaussian beam. The form of the higher order solutions, however, depend on the symmetry of the problem. For the case of cartesian symmetry, they are given by the Hermite-Gauss modes, while for cylindrical symmetry, they are given by Laguerre-Gauss modes [16]. In both cases, the modes are defined by two
integer indices $n$ and $m$. Moreover, these modes represent an orthonormal and complete basis in cartesian and cylindrical coordinates, respectively. Without any loss of generality, then we can define $\psi_{nm}(x, \omega)$ to be one of such paraxial eigenmodes. Any more general paraxial field can be then represented as a suitable superposition of these modes.

Let us then define the state of the field corresponding in having a single photon in one of these modes as follows:

$$|n, m, \omega, \lambda\rangle = \int d^2x \psi_{nm}(x, \omega)|x, \omega, \lambda\rangle,$$  \hspace{1cm} (11.29)

in such a way that

$$\langle y, \Omega, \mu|n, m, \omega, \lambda\rangle = \psi_{nm}(y, \omega)\delta_{\lambda\mu}\delta(\omega - \Omega).$$  \hspace{1cm} (11.30)

Analogously to what we did in Eq. \ref{eq:11.26}, we can define the mode operators as follows:

$$|n, m, \omega, \lambda\rangle = \hat{a}_{nm\lambda}(\omega)|0\rangle.$$  \hspace{1cm} (11.31)

It is then not difficult to prove that the following relations hold:

$$\hat{a}_{nm\lambda}(\omega) = \int d^2x \psi_{nm}^*(x, \omega)\hat{a}_\lambda(x, \omega),$$  \hspace{1cm} (11.32a)

$$\hat{a}_{nm\lambda}^+(\omega) = \int d^2x \psi_{nm}(x, \omega)\hat{a}_\lambda^+(x, \omega),$$  \hspace{1cm} (11.32b)

for the mode operators and

$$\hat{a}_\lambda(x, \omega) = \sum_{n,m} \psi_{nm}(x, \omega)\hat{a}_{nm\lambda}(\omega),$$  \hspace{1cm} (11.33a)

$$\hat{a}_\lambda^+(x, \omega) = \sum_{n,m} \psi_{nm}^*(x, \omega)\hat{a}_{nm\lambda}^+(\omega),$$  \hspace{1cm} (11.33b)

for the correspondent spatial operators.

12 Bibliography

References


REFERENCES


