Chapter 2: Quantization of the EM-Field and Photons as fundamental Excitations of its Modes

In this chapter we will develop the fundamentals of quantum-electrodynamics (QET), i.e. the generalization Maxwell's Equations, which naturally lead to the concept of the modes and then to photons, which populate these individual modes. They allows us to describe many quantum optical effects. We look into multi-mode, i.e. entangled, excitations later.

This process of quantization if often termed the “construction of the laws of QET”. This term is somewhat misleading; in reality, the process is educated guesswork, which combines three trains of thought:

- the compatibility requirement: the classical electrodynamic equations must retain their validity as an approximation to the new governing equations of QET,
- the ingenious idea: we follow the same approach, that links classical with quantum mechanics; namely we start from a canonical formulation of Maxwell’s Equations, including a classic Hamiltonian and canonical position and momenta, which we then just treat as operators. These classical quantities are constructed in a way, which leads to certain exchange rules, termed "Poisson Brackets {"}, which carry over to the operator regime as commutation equations.
- It must be validated by experiments, which reach beyond the approximation of wave physics (and has been done so frequently).

While this process seems "logical" in hindsight, this is still just guesswork. The validity of the construct needs to be tested experimentally (and as far as we know, it has passed every major test ever since its formulation).

2.1 Maxwell's Equation in Canonical Formulation

Maxwell's Equations can be written as the evolution equation to the Lagrangian density:

\[ \mathcal{L}(\phi, \dot{\phi}, A, \dot{A}) = \frac{\varepsilon_0}{2} E^2(r, t) - \frac{1}{2\mu} B^2(r, t) \]

where we have assumed free space propagation, i.e.

\[ j = 0 \quad \rho = 0 \]

And we have written the Lagrangian density in terms of the scalar potential \( \phi \) and the vector potential \( \mathbf{A} \), which generate the fields \( \mathbf{B} \) and \( \mathbf{E} \). For the sake of simplicity, we adopt Coulomb (or radiation gauge)

\[ \nabla \cdot \mathbf{A} = 0 \quad \phi = 0 \]

Then the relation take the simple form

\[ \mathbf{E} = -\frac{\partial \mathbf{A}(r, t)}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A}(r, t) \]

Maxwell's Equations can be obtained from the Lagrangian density by application of the Euler-Lagrange-Equations

\[ \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\phi}} - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \quad \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{A}_j} - \frac{\delta \mathcal{L}}{\delta A_j} = 0 \]

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\[ \Pi_\Phi = \frac{\delta L}{\delta \dot{\Phi}} = 0 \quad \Pi_A = \frac{\delta L}{\delta \dot{A}} = \epsilon_0 A \]

Where \( \Pi_\Phi \) and \( \Pi_A \) are the canonical momenta of \( \Phi \) and \( A \) respectively. By this construction, the canonical coordinates and canonical momenta automatically fulfill a set of commutation relations (not shown here), termed Poisson-Bracket relations.

The resulting Maxwell-Equations can be reformulated as the wave equation

\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0 \quad \epsilon_0 \mu_0 = c^{-2} \]

Each solution to this equation (i.e. each EM-field) can then be written as a superposition of plane waves

\[ A(r, t) = \sum_{\mathbf{k}} \int \frac{dk_x dk_y dk_z}{(2\pi)^32\omega_k} A_\mathbf{j}(\mathbf{k}) \epsilon_\mathbf{j}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \]

With the dispersion relation

\[ \omega_k^2 = k_x^2 + k_y^2 + k_z^2 \quad \omega_k = \pm \sqrt{k_x^2 + k_y^2 + k_z^2} \]

We can now construct the classical Hamiltonian Density by executing a Legendre transformation with respect to the dynamical variables \( \frac{\partial \Phi}{\partial t} \) and \( \frac{\partial A}{\partial t} \). We arrive at:

\[ \mathcal{H} = \Pi_\Phi \dot{\Phi} + \Pi_A \dot{A} - L = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \]

Which is (somewhat unsurprisingly) the energy density of the electromagnetic field, which we could have guessed. But, we would have not gotten the definition of the canonical momenta and positions from just guessing the Hamiltonian density. This is however an important ingredient in the quantization process, as they are crucial in the definition of observables to the system.

### 2.2 Field Quantization in Space

Now we quantize the electromagnetic field by adding a "hat" to all fields. I.e. we promote them from scalar (or vectorial) fields to operator fields. By construction, these obey certain commutation relations:

\[ \left[ \hat{A}^i(r, t), \hat{A}^k(r', t) \right] = 0 \]

\[ \left[ \hat{\Pi}_A^i(r, t), \hat{\Pi}_A^k(r', t) \right] = 0 \]

We will later see that these have a meaning beyond pure mathematics: it means that one can make independent measurements of the quantum field and independent measurements of its momentum (i.e. the \( E \) and \( B \) field) at two different points in space at arbitrary order, without mutual influence of the result.

As opposed to the classical theory, we however also get the following mixed commutation relation

\[ \left[ \hat{A}^i(r, t), \hat{\Pi}_A^j(r', t) \right] = i\hbar \delta_{ij}(r - r') \]

\[ \hat{A}^i = -\hat{A}^i \]

\[ \Delta_{ij}(r) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} (\delta_{ij} - \frac{k_i k_j}{k^2}) \]

Where the second equations comes into play, due to the relativistic nature of the fields and the third term is basically an ordinary \( \delta \)-function, which is corrected for the divergence-free nature of the EM-
field (i.e. that we have only two-polarizations for three spatial degrees of freedom). $k$ is the wave-vector of a plane wave, which we will introduce shortly.

This means that we cannot measure the same components of the field and its momentum independently at the same point in space and time. If you measure both, its result will depend on the order of the measurement.

2.3 Introduction of Quantum Plane Waves Modes

We can, of course decompose each quantum field into quantum plane waves. We will later see that due to the construction of the quantum fields, these quantum plane waves are eigenstates of the Hamiltonian-Operator of the system and thus remain shape invariant (expect for a phase term) under the evolution of time. They also have, due to the construction, the same spatial form and dispersion relation as the classical plane waves.

$$
\tilde{A}(r, t) = \sum_{\lambda} \int \frac{d k}{(2\pi)^3} \sqrt{\frac{\hbar}{\epsilon_0}} \left\{ \epsilon(k) \hat{a}_\lambda(k) u(k, r, t) + c. c. \right\}
$$

$$
u(k, r, t) = \frac{e^{i(kr - \omega t)}}{\sqrt{(2\pi)^3 2\omega k}}
$$

This means

- The only thing which is changed in QED is that the field strength in each plane wave mode is now a not a scalar number but an operator in itself
- Plane wave are still invariant modes of the system.
- The nature of polarization does not change

As we will later mostly just look into plane waves, it now makes sense to derive commutation relations for their operators. These are

$$
[\hat{a}_\lambda(k), \hat{a}_{\lambda'}^+(k')] = \delta_{\lambda\lambda'} \delta(k - k')
$$

$$
[\hat{a}_\lambda(k), \hat{a}_{\lambda'}(k')] = [\hat{a}_{\lambda'}^+(k), \hat{a}_\lambda^+(k')] = 0
$$

Which means that the state of any plane wave can be determined independently from the state of any other plane wave, expect for the state of a plane wave and it’s conjugate. To move ahead somewhat: you cannot determine the field and its derivative aka. the electric and the magnetic field of one mode (think about measuring currents and voltages in Electronics --> both measurements necessarily influence each other!!)

In order to derive these equations we have made use of the scalar product, which is needed to decompose the state of the quantum field into the appropriate modes. This modal(!) scalar product is defined as:

$$
(\phi, \psi) = i \int dr (\phi^* \partial_t \psi - (\partial_t \phi)^* \psi)
$$

This scalar product has some important features:

- It is a mathematical representation of the continuity equation and thus time-independent
- It defines a complete and orthogonal set of modes (this will not be derived here).
- This set is (by construction) the set of plane waves

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Using this scalar product we can invert the equation between $\hat{A}$ and $\hat{a}_j$, namely (see seminar for complete derivation)

$$\hat{a}_\lambda(k) = \frac{\sqrt{\varepsilon_0}}{\hbar} \left( u(k; r, t) \epsilon_\lambda(\hat{A}(r, t)) \right)$$

This equation gives us a recipe on how we can decompose any field into plane wave modes. Note that we have left out the +c.c for notational brevity; these however just end up as +0 in the integrals. In this derivation we have also implicitly proven, that the modes are orthogonal with respect to each other and that they are complete.

### 2.4 Non Plane-Wave-Modes

Of course, plane waves are by no means the only set of modes that can be used to describe the field. As they are, however, complete we can construct any basis set $v(k; r, t)$ from a superposition of them, i.e.

$$v(k; r, t) = \int dk V(k, \kappa) u(k; r, t)$$

Note that $\kappa$ is now any set of indices, which enumerates the new basis set and $V(k, \kappa)$ is a unitary matrix, i.e. $V^* = V^{-1}$. As with the plane wave modes above, they share the same relationship with the operator field $\hat{A}$ as the plane wave operators, i.e. the scalar product works the same way for all sets of eigenmodes:

$$\hat{b}_\lambda'(k) = \frac{\sqrt{\varepsilon_0}}{\hbar} \left( v(k) \epsilon_\lambda(\hat{A}) \right)$$

This relation can also be evaluated in terms of the plane wave operators $\hat{a}_\lambda(k)$, this yields:

$$\hat{b}_\lambda'(k) = \sum_\lambda \int dk \left[ \epsilon^*_\lambda(\kappa) \epsilon_\lambda(k) (v(k), u(k)) \hat{a}_\lambda(k) + \epsilon^*_\lambda(\kappa) \epsilon_\lambda(k) (v(k), u^*(k)) \hat{a}_\lambda^*(k) \right]$$

Note that by the construction the modes $\hat{b}_\lambda(k)$ have the same commutation relation as $\hat{a}_\lambda(k)$. Also note that we have been somewhat sloppy in the transformation of the polarization eigenmodes, but these can be included in the equations in a straightforward manner.

It is indeed useful as plane waves are kind of non-physical. The have unbounded energy and they are of infinite extent. Particularity for experiments it often makes sense to describe the field in something more...close to reality.

#### 2.4.1 Example 1: Gaussian Modes

For the introduction of Gaussian modes we will first neglect polarization altogether (assuming, i.e. Linear polarization, which is in itself an approximation, as the different plane waves that compose a Gaussian have different polarization states) and then we assume paraxiality, i.e. the beam diameter is much larger than the wavelength of light. We will also assume that the field is harmonic with a frequency of $\omega_0$ (i.e. it is of a single frequency) and that its propagation direction is mostly along the $z$-axis. Thus

$$k_z \approx k \left( 1 - \frac{k_x^2 + k_y^2}{k^2} \right) = \frac{\omega}{c}$$

Under these assumptions, the Gaussian Mode takes the form

$$\hat{A}_0(r, t) = \int d\omega (\tilde{A}(\omega) A_0(r, \omega) e^{i\omega t} + c.c)$$

$$A_0(r, \omega) = \frac{4\pi}{\sqrt{\varepsilon_0 s^2(z)}} \exp \left( ik_z \frac{x^2 + y^2}{s^2(z)} \right)$$
\[ s^2(z) = w_0^2 + \frac{2lz}{k} \]

One can indeed show, that these transverse modal fields are minimum uncertainty localized transverse modes, i.e. they are the modes which for a given diameter have the least divergence:

\[ \Delta k \Delta x = \frac{1}{4\pi} \int dk |f(k)|^2 \rightarrow w_0 NA = \frac{\lambda}{\pi} \]

They are therefore well suited for long-range communication, as they require the smallest telescopes.

2.4.1 Example 1: Gauss-Laguerre Modes

We can extend on some of the relations found for the Gaussian Modes and extend these onto a complete set of Eigenmodes with rotational symmetry, i.e. we introduce

\[ x = r \cos \varphi \quad y = r \sin \varphi \quad \sigma = x + iy \]

The Gauss-Laguerre modes have the form

\[ A_{lm}(r, \omega) = \frac{4\pi(-1)^{l+m}l!}{\sqrt{\pi} w_0^2 s^2(z)} |m| e^{im\varphi} L^m_l \left( \frac{r^2}{s^2(z)} \right) e^{i kr - \frac{r^2}{s^2(z)}} + c.c \]

\[ u_{lm}^{LG}(k, r, t) = A_{lm}(r, \omega) e^{i\omega t} \]

\[ \Rightarrow \hat{A}(r, t) = \sqrt{\frac{\pi w_0}{\hbar}} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \int dk \hat{a}_{lm}(k) u_{lm}^{LG}(k, r, t) + c.c \]

Note that \( s(z) \) was defined in the last chapter. Also note that the \( \varphi \)-dependency is exclusively a phase term \( \sim \exp(-im\varphi) \). Thus one can easily see that these modes are eigenfunctions to the operator, which measures the z-coordinate of the angular orbital momentum \( \hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial \varphi} \) with the Eigenvalue \( \hbar m \).

\[ \hat{L}_z u_{lm}^{LG}(k, r, t) = m\hbar u_{lm}^{LG}(k, r, t) \]

These beams thus carry a quantized and measurable orbital angular momentum. As this is a discrete quantity it can be used conveniently to transport information with more than a bit per photon (i.e. a so-called qdit, compare to the concept of QAM for example in classical communication). Also note that this information transfer is quite robust: the angular momentum is a compatible measurable to...
both the direction \((k_0)\) of the beam, its frequency \((\omega)\) as well as its overall impulse \((l)\). Propagation through air typically induces perturbation along \(k_0\) and \(l\) but very little on \(m\). Information encoded in these modes is thus also robust.

**2.5 Polarization modes**

So far, we have pretty much ignored the polarization aspects of the modes. We shall now have a closer look at these. As with classical EM-theory these can be represented with Jones Vectors

\[
\psi(k) = \begin{bmatrix} \psi_1(k) \\ \psi_2(k) \end{bmatrix}
\]

Where we have assumed, without loss of generality, that \(k = k_0 \hat{e}_z\). Then we can find a few single basis-vector systems, in which we can describe the polarization state of light:

- **linear HV**: \(\psi = h \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} = h|k\rangle + v|\psi\rangle\)
- **linear diagonal**: \(\psi = u \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + d \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = u|u\rangle + d|d\rangle\)
- **linear**: \(\psi = l_1 \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix} + l_2 \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix} = l_1|l_1\varphi\rangle + l_2|l_2\varphi\rangle\)
- **circular**: \(\psi = l \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} + r \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} = l|l\rangle + r|r\rangle\)

**2.6 Quantization of the Hamiltonian**

Now that we have introduced the quantum modal operators \(\hat{a}_\lambda(k)\) it is quite straightforward to show that the QET analogue of the Hamilton-Operator is

\[
\hat{H} = \sum_\lambda \int dk \frac{\hbar \omega(k)}{2} \left( \hat{a}_\lambda(k) \hat{a}_\lambda^\dagger(k) + \hat{a}_\lambda^\dagger(k) \hat{a}_\lambda(k) \right)
\]

\[
= \sum_\lambda \int dk \ h\omega(k) \left( \hat{a}_\lambda^\dagger(k) \hat{a}_\lambda(k) + \hat{e}_\lambda(k) \right)
\]

\[
= \sum_\lambda \int dk \ \hat{H}_\lambda(k)
\]

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The Hamiltonian has the following commutation relations, i.e. you can’t measure the state of a single mode, without interfering with the energy of this state.

\[
\begin{align*}
[\hat{H}, \hat{a}_\lambda(k)] &= -\hbar \omega \hat{a}_\lambda(k) \\
[\hat{H}, \hat{a}_\lambda^\dagger(k)] &= \hbar \omega \hat{a}_\lambda^\dagger(k)
\end{align*}
\]

Let’s assume that we have found eigenstates for the Hamiltonian, such that

\[
\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle
\]

Then these can be combined with the operator commutation equation

\[
\begin{align*}
\hat{H}\hat{a}_\lambda(k)|\psi_n\rangle &= \hat{a}_\lambda(k)\hat{H}|\psi_n\rangle - \hbar \omega \hat{a}_\lambda(k)|\psi_n\rangle \\
&= (E_n - \hbar \omega)\hat{a}_\lambda(k)|\psi_n\rangle \\
\hat{H}\hat{a}_\lambda^\dagger(k)|\psi_n\rangle &= (E_n + \hbar \omega)\hat{a}_\lambda^\dagger(k)|\psi_n\rangle
\end{align*}
\]

This means that \(\hat{a}_\lambda(k)\) reduces the eigenvalue (i.e. energy) of the state \(|\psi_n\rangle\) by a certain quantity but it’s still an eigenstate. Same for \(\hat{a}_\lambda^\dagger(k)\) just that it increases the eigenvalue (i.e. energy). As the Eigenvalues of \(\hat{H}\) must be bound from below (it’s a energy after all and negative energy is kind of hard to come by) there should be a ground state for which

\[
\hat{H}|\psi_0\rangle = 0 \quad \forall \lambda, k
\]

This is called the quantum-vacuum state and is will be denoted as \(|0\rangle\). However, if one calculates its energy one gets:

\[
\hat{H}|\psi_0\rangle = \sum_\lambda \int dk \ h \omega(k) \ \hat{a}_\lambda^\dagger(k)\hat{a}_\lambda(k)|\psi_0\rangle = 2 \int dk \delta(k - k') |\psi_0\rangle
\]

This term is the quantum vacuum energy \(E\). It diverges and must be removed for all practical calculation of the energy. It’s however not unphysical. It leads e.g. to the Lamb-Shift, the Casimir-Force, and the Quantum-Unruh-Effect (dynamical Casimir Effect).

The divergence does however occur for two reasons. (1) Because of the infinite sum and (2) because of the \(\delta\)-function. It turns out the latter is a problem of the temporal structure of the modes, which we have ignored so far, i.e. we have assumed them to be ill-behaved, infinite, naughty plane waves. But we can fix this. However, we first introduce (plane-wave) Fock-States \(|n_{k,\lambda}\rangle\) for every mode denoted by \(k\) and \(\lambda\) by repeatedly applying \(\hat{a}_\lambda^\dagger(k)\) \(n\) times to \(|\psi_0\rangle\). These then have the relative energy:

\[
E_n(k) = h \omega(k)n
\]

We also define the number-operator

\[
\hat{n} = \sum_\lambda \int dk \ \hat{a}_\lambda^\dagger(k)\hat{a}_\lambda(k)
\]

Which has a well-defined meaning for Fock-States \(|n_{k,\lambda}\rangle\)

\[
\hat{n}|n_{k,\lambda}\rangle = n|n_{k,\lambda}\rangle
\]

As \(\hat{a}_\lambda^\dagger(k)\) and \(\hat{a}_\lambda(k)\) can be used to move us up and down the ladder of Fock-States, we thus call them ladder-operators or creation and annihilation operators.
The plane-wave Fock-States still have the above-mentioned naughtiness, i.e. they are not normalizable

\[ \langle 1_{k\lambda} | 1_{k'\lambda'} \rangle = \langle 0_{k\lambda} | \hat{a}_{k\lambda}^\dagger(\mathbf{k}) \hat{a}_{k'\lambda}(\mathbf{k}) | 0_{k'\lambda'} \rangle = \langle 0_{k\lambda} | \hat{a}_{k\lambda}^\dagger(\mathbf{k}) \hat{a}_{k'\lambda}(\mathbf{k}) | 0_{k'\lambda'} \rangle + \delta_{ij} \delta(\mathbf{k} - \mathbf{k}') = \delta_{ij} \delta(\mathbf{k} - \mathbf{k}') \]

We’ll now simply make a Basis transformation into a set of modes \( \nu_{j,\lambda} \), which are centered around a particular wave-vector \( \mathbf{k}_j \) and which themselves form an orthonormal basis. We can then decompose the quantum field \( \hat{A} \) into these modes using a Bogolioubov transformation

\[ \hat{b}_{j\lambda} = \sqrt{\frac{\epsilon_0}{\hbar}} (\epsilon_{j\lambda} \nu_{j\lambda}, \hat{A}) = \int d\mathbf{k} \left( \alpha_j(\mathbf{k}) \hat{a}_{\lambda}(\mathbf{k}) + \beta_j(\mathbf{k}) \hat{a}_{\lambda}^\dagger(\mathbf{k}) \right) \]

For the sake of simplicity we can assume that \( \beta_j(\mathbf{k}) = 0 \) and of course we know \( \int d\mathbf{k} |\alpha_j(\mathbf{k})|^2 = 1 \).

By construction the operators, fulfil the commutation relations

\[ [\hat{b}_{j\lambda}, \hat{b}_{j'\lambda'}^\dagger] = \delta_{j,j'} \delta_{\lambda,\lambda'} \]

We can then now attempt to construct a set of Fock-Modes for this basis starting from the first mode

\[ |1_{j\lambda}\rangle = \hat{b}_{j\lambda}^\dagger |0\rangle = \int d\mathbf{k} \alpha_j^\dagger(\mathbf{k}) \hat{a}_{\lambda}(\mathbf{k}) |0\rangle \]

Let’s test these for normalizability:

\[ \langle 1_{j\lambda} | 1_{j\lambda} \rangle = \int d\mathbf{k} d\mathbf{k}' \alpha_j(\mathbf{k})^\dagger \alpha_j(\mathbf{k}')^\dagger |0\rangle \langle 0 | \hat{a}_{\lambda}(\mathbf{k}) \hat{a}_{\lambda}^\dagger(\mathbf{k}') |0\rangle = \int d\mathbf{k} d\mathbf{k}' \alpha_j(\mathbf{k})^\dagger \alpha_j(\mathbf{k}')^\dagger |0\rangle \langle 0 | \hat{a}_{\lambda}(\mathbf{k}) \hat{a}_{\lambda}^\dagger(\mathbf{k}') + \delta_{\lambda,\lambda'} \delta(\mathbf{k} - \mathbf{k}') |0\rangle = \delta_{\lambda,\lambda'} \int d\mathbf{k} d\mathbf{k}' \alpha_j(\mathbf{k})^\dagger \alpha_j(\mathbf{k}')^\dagger |0\rangle \langle 0 | \hat{a}_{\lambda}(\mathbf{k}) \hat{a}_{\lambda}^\dagger(\mathbf{k}') = \delta_{\lambda,\lambda'} \int d\mathbf{k} \alpha_j(\mathbf{k}) \alpha_j^\dagger(\mathbf{k}) = \delta_{\lambda,\lambda'} \]

This is now well-behaved! Yay. Keep in mind that the function \( \alpha_j(\mathbf{k}) \) may be very localized, such that from an experimental point of view here is very little difference to a plane wave here. We’ll therefore in the future often forget the difference between \( \hat{b}_{j\lambda} \) and \( \hat{a}_{\lambda}(\mathbf{k}) \). We will later do the same for the temporal structure of the mode; let’s call this “modal doublethink”. If you are really worried about this, then you are a good mathematician.

Let us now use the normalized mode operators to properly normalize their respective Fock-Modes with \( n > 1 \), which we could not do previously. The result of this process is:

\[ |n_{j\lambda}\rangle = \frac{1}{\sqrt{n!}} (\hat{b}_{j\lambda}^\dagger)^n |0\rangle \]

\[ \hat{b}_{j\lambda}^\dagger |n_{j\lambda}\rangle = \sqrt{n} |n-1_{j\lambda}\rangle \]

\[ \hat{b}_{j\lambda} |n_{j\lambda}\rangle = \sqrt{n+1} |n+1_{j\lambda}\rangle \]

\[ \hat{N}_{j\lambda} |n_{j\lambda}\rangle = \hat{b}_{j\lambda}^\dagger \hat{b}_{j\lambda} |n_{j\lambda}\rangle = n |n_{j\lambda}\rangle \]

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The complete Hamiltonian can then be written as

\[ \hat{H} = \sum_\lambda \int d\omega \hbar \omega \hat{n}_{j,\lambda} \]

This has a quite straightforward interpretation:

- Each mode \( f_{j,\lambda} \) is filled with \( n \) particles, if it is in state \( |n_{j,\lambda}\rangle \)
- The \( \hat{b}_{j,\lambda}^\dagger \) operator creates one particle in mode \( f_{j,\lambda} \)
- The \( \hat{b}_{j,\lambda} \) operator destroys one particle in mode \( f_{j,\lambda} \)
- Each of these particles is called a photon.
- Each photon has a set of observables.

### 2.7 Coherent States

In the last chapter we have introduced Fock-states, which are eigenstates to both, the Hamilton-Operator (i.e. the energy of the system) as well as the photon number operator. We have also seen that they can be created from the quantum vacuum state \( |0\rangle \) by repeated application of the photon creation operator \( \hat{a}^\dagger \).

Fock-states, are, however, fairly rare in nature (in fact, Fock-states with large numbers of photons in any given mode are extremely hard to produce!). The deeper reason being, that a number of photons is typically produced in a random process, where a large number of emitters is each emitting a photon with a certain non-unity chance (the prime example is the amplification process in a laser). This quantum randomness does naturally lead to quantum uncertainty in the photon number of a so-produced state of light (Given that the emission process’ probability does not change over time and not much in dependence of its history, one can already kind of guess that the resulting state of the field should have a Poisson-distribution of the photon numbers).

We will nevertheless use the Fock-states, as we have seen that they are a complete set of eigenstates to the state of any given mode. We will construct a new set of modes from a superposition of these Fock-states, by application of a superposition of creation/annihilation-operators to the vacuum state:

\[ \hat{D}(\alpha) = \exp\{a\hat{a}^\dagger - a^*\hat{a}\} \]

Where \( \alpha \) is a complex number and \( \hat{D}(\alpha) \) is a unitary operator (we will see in the next chapter, that this is a necessary requirement for such a generation operator). In this case \( \hat{D}(\alpha) \) is called the \"Glauber displacement operator\". Unitarity can be easily proven by checking the following relations:

\[ \hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha) = \]

Let’s now rewrite the operator, using the commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\)

\[ \hat{D}(\alpha) = \exp\left\{a\hat{a}^\dagger - a^*\hat{a} - \frac{1}{2}[a\hat{a}^\dagger, -a^*\hat{a}] + \frac{1}{2}[a\hat{a}^\dagger, -a^*\hat{a}]\right\} \]

\[ = \exp\left\{-\frac{|\alpha|^2}{2}\right\} \exp\left\{a\hat{a}^\dagger - a^*\hat{a} - \frac{1}{2}[a\hat{a}^\dagger, -a^*\hat{a}]\right\} \]

\[ = \exp\left\{-\frac{|\alpha|^2}{2}\right\} \exp\{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]\} \]

\[ = \exp\left\{-\frac{|\alpha|^2}{2}\right\} \exp\{\hat{A}\} \exp\{\hat{B}\} \]

\[ \Leftrightarrow [\hat{A}, \hat{B}] = [\hat{A}, \hat{B}] = 0 \text{ with } \hat{A} = a\hat{a}^\dagger, \hat{B} = -a^*\hat{a} \]
\[= \exp\left\{-\frac{|\alpha|^2}{2}\right\}\exp\{\alpha \hat{a}^\dagger\}\exp\{-\alpha^* \hat{a}\}\]

Then we can apply this reformulated version of the Glauber-Operator easily on the Vacuum-State

\[|\alpha\rangle = \hat{D}(\alpha)|0\rangle = \exp\left\{-\frac{|\alpha|^2}{2}\right\}\exp\{\alpha \hat{a}^\dagger\}|0\rangle = \exp\left\{-\frac{|\alpha|^2}{2}\right\}\sum_n \frac{\alpha^n (\hat{a}^\dagger)^n}{n!}|n\rangle\]

This means, that the field is in a superposition of Fock-States and the probability \(P(n)\) (amplitude square!!!) of finding the field in an \(|n\rangle\) state is given by the Poisson distribution

\[P(n) = |\langle n|\hat{D}(\alpha)|0\rangle|^2 = \exp\left\{-\frac{|\alpha|^2}{2}\right\} \left(|\alpha|^2\right)^n \frac{\sqrt{n!}}{\sqrt{n^n}} = P_{\text{Poisson}}(n, |\alpha|^2)\]

From probability theory we know, that a series of Poisson-distributed events is maximally random, i.e. the occurrence of an event (i.e. the appearance of a photon) at any given point in time in a certain mode does by now means make the time of appearance of another photon more or less probable. In this respect, coherent states have no memory, photons are neither bunched, nor anti-bunched.

**FIG. 2: PHOTON-NUMBER PROBABILITIES OF TWO DIFFERENT COHERENT STATES.**

We can quite easily find the expectation value and variance of the photon number operator:

\[\langle \hat{n} \rangle = \langle \alpha |\hat{n}|\alpha \rangle = \exp\{-|\alpha|^2\} \sum_{n,n'} \frac{\alpha^n \alpha'^{n'}}{\sqrt{n!} \sqrt{n'!}} \langle n'|\hat{n}|n \rangle = \exp\{-|\alpha|^2\} \sum_n \frac{|\alpha|^{2n}}{n!} n = |\alpha|^2\]

\[(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2\]

All notes subject to change, no guarantee to correctness, corrections welcome.

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This has a few major ramifications. Ordinary light sources emit states of light, which do NOT have a defined number of photons. If you measure the energy you get a so-called "shot-noise" even for a perfect detector, which defines the measurement accuracy. Examples:

- 10 µW Signal on a 10 GHz Communication Channel $\rightarrow$ $10^{-15} \text{J}$ per time slot $\rightarrow$ roughly $10^{-18} \text{J}$ per photon for light with a wavelength of 1000 nm $\rightarrow$ 1000 Photons and a short noise floor of $\sqrt{1000} \approx 30$ photons $\rightarrow$ SNR of roughly 30; no more than $\log_2 \text{SNR} \approx 5$ bits per time slot
- Low-Light image with roughly 10 Photons per pixel $\rightarrow$ 3 Photons Shot Noise $\rightarrow$ 30% Noise floor

Both Communication- as well as Imaging can profit from the usage of Fock-States. Particularly the latter is one a goal of Quantum-Imaging and a hot topic in research.

Let’s now proceed to a few more properties of coherent states. First they are robust against mixing (i.e. amplification and damping):

$$\mathcal{D}(\beta)|\alpha\rangle = |\alpha + \beta\rangle$$

Coherent states are also complete:

$$\int d^2 \alpha \frac{1}{\pi} |\alpha\rangle\langle \alpha| = 1$$

They are also eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

We shall later see, that the time evolution of the any state is given by the application of the time evolution operator $\exp \left(-\frac{i}{\hbar} \hat{H} t \right)$, in this case this yields:

$$\exp \left(-\frac{i}{\hbar} \hat{H} t \right) |\alpha\rangle = \exp \left(-\frac{i}{\hbar} \hat{H} t \right) \exp \left\{-\frac{|\alpha|^2}{2} \right\} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \exp \left\{-\frac{|\alpha|^2}{2} \right\} \sum_n \frac{\alpha^n}{\sqrt{n!}} \exp \left(-\frac{i}{\hbar} \hat{H} t \right) |n\rangle$$

$$= \exp \left\{-\frac{|\alpha|^2}{2} \right\} \sum_n \frac{\alpha^n}{\sqrt{n!}} \exp \{-i\omega nt\} |n\rangle \quad \text{with} \quad \hat{H} |n\rangle = \hbar \omega n |n\rangle$$

$$= \sum_n \exp \left\{-\frac{|\alpha|^2}{2} \right\} \frac{\alpha \exp(-i\omega t)^n}{\sqrt{n!}} |n\rangle$$

The coherent states do thus have a time evolution, which can be represented by a rotation in the $\alpha$-plane.
2.8 Time-Evolution

The field of science is called Quantum Electro DYNAMICS. So far we have only looked into Quantum Electro Statics and have not looked into the evolution of the fields. We will now take a look at the free evolution of the field and find out, how this is related to the Hamiltonian-Operator.

We will then find, that this concept can indeed be expanded into arbitrary Interaction-Hamiltonians, which may describe real-world optical elements, such as beam-splitters, loss-elements and the like.

In the Heisenberg picture an arbitrary Hermitian operator $\hat{A}$ evolves in time under the influence of the time evolution operator $\hat{U}(t)$

$$\hat{U}(t) = \exp \left\{ -\frac{i}{\hbar} \hat{H} t \right\} $$

$$\frac{d\hat{U}(t)}{dt} = -\frac{i}{\hbar} \hat{H} \hat{U}(t) $$

$$\hat{A}(t) = \hat{U}(t)\hat{A}(t=0)\hat{U}^+(t) $$

$$\frac{d\hat{A}(t)}{dt} = i \left[ \frac{\hat{H}}{\hbar}, \hat{A} \right] + \frac{\partial\hat{A}}{\partial t}$$

We can also use these relations to derive a version of the famous Baker-Canbell-Hausdorff-Theorem, that we will later need:

$$e^{\mu\hat{B}}e^{-\mu\hat{B}} = \hat{A} + \mu [\hat{B}, \hat{A}] + \frac{\mu^2}{2} [\hat{B}, [\hat{B}, \hat{A}]] + \cdots$$

In a next step we will see, how this impacts the modal creation and annihilation operators $\hat{\alpha}_\lambda(\mathbf{k})$, which are themselves not time-dependent.

$$\frac{d\hat{\alpha}_\lambda(\mathbf{k})}{dt} = i \left[ \frac{\hat{H}}{\hbar}, \hat{\alpha}_\lambda(\mathbf{k}) \right]$$

$$\Rightarrow \hat{\alpha}_\lambda(\mathbf{k}, t) = \hat{\alpha}_\lambda(\mathbf{k}) \exp\{-i\omega(\mathbf{k})t\}$$

The time depended wave annihilation $\hat{\alpha}_\lambda(t)$ and photon number operators $\hat{n}(t)$ of the entire field are then:
\[
\hat{a}_\lambda(t) = \int dk \hat{a}_\lambda(k) \exp\{-i\omega(k) t\}
\]
\[
\hat{n}(t) = \hat{a}_\lambda^\dagger(t) \hat{a}_\lambda(t) = \int \int dk dk' \hat{a}_\lambda^\dagger(k') \hat{a}_\lambda^\dagger(k) \exp\{-i(\omega(k') - \omega(k)) t\}
\]

This is basically the same as the Bogolioubov transformations defined a few chapters earlier. This means that the time evolution of a state can be modelled as a transformation of the basis vectors with a very peculiar set of basis functions.

### 2.9 Temporally Localized Wavepackets

We can now use this knowledge to define temporally localized wave-packets using

\[
|1_{j\lambda}\rangle = \int dk \alpha_j(k) \hat{a}_\lambda(k) |0\rangle
\]

The number density at any time is then:

\[
\langle 1_{j\lambda} | \hat{n}_\lambda(t) | 1_{j\lambda}\rangle = \int dk \alpha_j(k) \exp\{-i\omega(k) t\}
\]

Some important wave-packet outlines are:

- **Lorentzian**: \(\alpha_L(k) = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\gamma}}{\sqrt{\gamma + i(\omega(k) - \omega_0)}}\)
- **Gaussian**: \(\alpha_G(k) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left\{ - \frac{(\omega(k) - \omega_0)^2}{4\sigma^2} \right\}\)
- **Rect**: \(\alpha_R(k) = \frac{1}{\sqrt{\Delta\omega}} \text{sinc}\left\{ \frac{\omega(k) - \omega_0}{\Delta\omega} \right\}^{1/2}\)

### 2.10 Mode-Transformation and Optical Elements

In chapter 2.8 we have seen, how the Hamiltonian defines the time evolution of the state operators \(\hat{a}\) and \(\hat{a}^\dagger\). We have also seen, that the effect is a multiplication with a phase, which is defined by the mode’s frequency and the time delay, just as we expect from classical optics (i.e. the state changes with the light’s frequency and different eigenstates are not mixed, e.g. this evolution is independent from the number of photons). We have also seen that the result of time evolution is a special Bogolioubov transformation; i.e. the action of time maps the state of the field at one instance in time \(t = t_1\) to the state of the field at a second instance of time by \(t = t_2\) by means of the special basis transformation.

In this chapter we want to generalize on this notion, using the idea, that lossless linear optical elements, i.e. phase shifters and beam splitters can be modelled (as in classical optics) as basis function transformations. We should therefore be able to model these with optical elements operators, which act on the field, similarly to the Hamiltonian operator.

These “interaction Hamiltonians” do, however, not act as a function of a continuous parameter (i.e. time) but they serve purely to map the state of light at the input of an element onto the state of light at the output of an element. There is quite a lot of modal doublethink involved here, too. Because the notion of a mode is now strictly applied only to one half of the \(\mathbb{R}^3\). With a different set on the other half. Alas.
Let’s begin with a simple phase shifter, which creates a phase shift by $\phi$ for a certain mode with index $j\lambda$. This has the same effect as the single mode Hamiltonian acting for a time of $\omega t = \phi$. We can thus describe the interaction Hamiltonian of a phase shifter by:

$$\hat{H}(\phi) = \hbar \phi \hat{a}_{j\lambda}^\dagger \hat{a}_{j\lambda}$$

And its action on the field by

$$\hat{U}(\phi) = \exp\left\{-i\phi \hat{a}_{j\lambda}^\dagger \hat{a}_{j\lambda}\right\}$$

This will induce the transformation

$$\begin{align*}
\hat{b}_{j\lambda} &= \hat{U}(\phi) \hat{a}_{j\lambda} \hat{U}^\dagger(\phi) \\
&= \exp\left\{-i\frac{\hbar}{\hbar} \hat{H}(\phi) \right\} \hat{a}_{j\lambda} \exp\left\{i\frac{\hbar}{\hbar} \hat{H}(\phi) \right\} \\
&= \hat{a}_{j\lambda} + \left(-i\frac{\hbar}{\hbar}\right) \left[\hat{H}(\phi), \hat{a}_{j\lambda}\right] + \frac{1}{2} \left(-i\frac{\hbar}{\hbar}\right)^2 \left[\hat{H}(\phi), \left[\hat{H}(\phi), \hat{a}_{j\lambda}\right]\right] + \cdots \\
&= \hat{a}_{j\lambda} + \left(-i\frac{\hbar}{\hbar}\right) \hat{a}_{j\lambda} + \frac{1}{2} \left(-i\frac{\hbar}{\hbar}\right)^2 \hat{a}_{j\lambda}^2 + \cdots \\
&= \exp\{-i\phi\} \hat{a}_{j\lambda}
\end{align*}$$

Where we have used the identity $\left[\hat{H}(\phi), \hat{a}_{j\lambda}\right]=\hbar \phi \hat{a}_{j\lambda}$ and the version of the BCH-theorem mentioned in the last chapter. The result is just as expected. The phase shifter adds a phase $\phi$ to the mode in question.

Next we will look into a beam splitter. The beam splitter mixes two modes. We thus guess the Hamiltonian to be

$$\hat{H}_{ji}(\zeta, \varphi) = \hbar \zeta (e^{i\varphi} \hat{a}_{j\lambda}^\dagger \hat{a}_{i\lambda} + e^{-i\varphi} \hat{a}_{i\lambda}^\dagger \hat{a}_{j\lambda})$$

We carry out the exact same derivation from above (although a bit more complex) and find
\[
\begin{bmatrix}
\hat{b}_\lambda
\\hat{\lambda}
\end{bmatrix} = \begin{pmatrix}
\cos \zeta & -i e^{i\varphi} \sin \zeta \\
-ie^{-i\varphi} \sin \zeta & \cos \zeta
\end{pmatrix} \begin{bmatrix}
\hat{a}_\lambda
\\hat{\lambda}
\end{bmatrix}
\]

These can also be two equal modes of different polarization, then \( j = l \) and \( \lambda = 1 \), where \( j \) and \( \lambda = 2 \) where \( l \).

If we combine both, by introducing a differential phase shift of \( \pm \varphi/2 \) in the first and second mode we get the matrix

\[
\tilde{U}(\phi, \zeta, \varphi) = \begin{pmatrix}
e^{i\phi/2} \cos \zeta & -ie^{i\varphi} \sin \zeta \\
-ie^{-i\varphi} \sin \zeta & e^{-i\phi/2} \cos \zeta
\end{pmatrix}
\]

Note that this is the most general Unitarian 2x2 matrix. I.e. ANY lossless, photon-number-conserving, linear mode transformation between pairs of modes can be written as such a matrix. This also means that a combination of beam-splitter and a phase shifter can create ANY possible lossless, linear, photon-number-conserving interaction between two modes there is.

The action of the multiple interaction Hamiltonians can of course be "stacked" to form complex optical elements. Here is a look at a Mach-Zender interferometer, i.e. a stacked BS-PS-BS. For brevity we assume it's balanced and all phase shift is in the bottom arm.

\[
\begin{bmatrix}
\hat{b}_1 \\
\hat{b}_2
\end{bmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
e^{i\phi} & 1
\end{pmatrix} \begin{bmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{bmatrix}
\]

\[
\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \Leftarrow \text{Photon number conservation}
\]

\[
\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2 = \cos \varphi (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) - i \sin \varphi (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \Leftarrow \text{Interference}
\]

As expected we see that the total number of photons is preserved but there is a shift of contrast, which is enacted by the phase shift \( \varphi \). We will later discuss this result extensively.

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**Fig. 6:** Feynman representation of a mode squeezing operator. The green interaction paths are most commonly realized by a three photon interaction, where the resulting photon is discarded.

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Some more notes on this topic:

- **An N-port interferometer** (i.e. an arbitrary \( n \times n \) Unitary Matrix) can also be constructed from a series of 2-port beam-splitters and phase shifters \( \rightarrow \) any linear optical element can be thought of as a (possibly very complicated) set of beam-splitters and phase -shifters

- Here we have **only considered photon-number-preserving interaction-Hamiltonians**, i.e. those that depend on a sum of creation-operators multiplied with annihilation operators, e.g. \( \hat{a}^\dagger \hat{b} \). One can also consider those of type \( \hat{a} \hat{b} \). These model the **simultaneous annihilation**
of two photons or their simultaneous creation. These lead to squeezed states of light. These operators can be implemented using nonlinear optics (e.g. sum frequency generation of a pump photon destroys a signal and an ideal photon at the same time), if some channels of light are ignored.

- Any quadratic Interaction Hamiltonian is considered to be linear. This is also true for squeezing operators, which require NLO to be implemented.

This concludes chapter 2. We have seen how the EM-field is quantized, how it is decomposed into modes, how we can transform between different sets of modes. We have looked into fundamental states of individual modes and came to identify these with the concept of photons. We have also see how these evolve over time and what simple optical elements do with them. This has naturally introduced us to the concept of mixed modes.

In the next chapters we will look in more detail on how one can create and measure the properties, which we have defined here, in detail and what happens, if we do no longer limit ourselves to single mode excitations, which will naturally introduce the concept of entanglement.