Chapter 2: Quantization of the Electromagnetic Field

In this chapter we will develop the fundamentals of quantum-electrodynamics (QET), i.e. the generalization Maxwell’s Equations, which naturally lead to the concept of the photons and allows us to describe all effects related to these.

This process is often termed the “construction of the laws of QET”. This term is somewhat misleading; in reality, the process is educated guesswork, which combines two trains of thought:

- the compatibility requirement: the classical electrodynamic equations must retain their validity as an approximation to the new governing equations of QET,
- the ingenious idea: we follow the same approach, that links classical with quantum mechanics; namely we start from a canonical formulation of Maxwell’s Equations, including a classic Hamiltonian and canonical position and momenta, which we then just treat as operators. These classical quantities are constructed in a way, which leads to certain exchange rules, termed “Poisson Brackets {}”, which carry over to the operator regime as commutation equations.

Note: While this process seems "logical" in hindsight, this is still just guesswork. The validity of the construct needs to be tested experimentally (and as far as we know, it has passed every major test ever since its formulation).

2.1 Maxwell's Equation in Canonical Formulation

Maxwell’s Equations can be written as the evolution equation to the Lagrangian density:

\[ \mathcal{L}(\phi, \dot{\phi}, A, \dot{A}) = \frac{\varepsilon_0}{2} E^2(r, t) - \frac{1}{2\mu} B^2(r, t) \]

where we have assumed free space propagation, i.e.

\[ \mathbf{j} = 0 \quad \rho = 0 \]

And we have written the Lagrangian density in terms of the scalar potential \( \phi \) and the vector potential \( A \), which generate the fields \( B \) and \( E \). For the sake of simplicity we adopt Coulomb (or radiation gauge)

\[ \nabla \cdot A = 0 \quad \phi = 0 \]

Then the relation take the simple form

\[ E = -\frac{\partial A(r, t)}{\partial t} \quad B = \nabla \times A(r, t) \]

Maxwell’s Equations can be retained from the Lagrangian density by application of the Euler-Lagrange-Equations

\[ \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\phi}} - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \quad \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{A}_j} - \frac{\delta \mathcal{L}}{\delta A_j} = 0 \]

\[ \Pi_\phi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = 0 \quad \Pi_A = \frac{\delta \mathcal{L}}{\delta \dot{A}} = \varepsilon_0 \dot{A} \]
Where $\Pi_\Phi$ and $\Pi_A$ are the canonical momenta of $\Phi$ and $A$ respectively. By this construction the canonical coordinates and canonical momenta automatically fulfil a set of commutation relations (not shown here).

The resulting Maxwell-Equations can be reformulated in the wave equation

$$\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0 \quad \epsilon_0 \mu_0 = c^{-2}$$

Each solution to this equation (i.e. each EM-field) can then be written as a superposition of plane waves

$$A(r, t) = \sum_\lambda \int_{\mathbb{R}^3} dk x dk_y dk_z \frac{1}{\sqrt{\epsilon_0 (2\pi)^3 2 \omega_k}} A_\lambda(k) e^{i(kr - \omega_k t)}$$

With the dispersion relation

$$\frac{\omega_k^2}{c^2} = k_x^2 + k_y^2 + k_z^2 \quad w_k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

We can now construct the classical Hamiltonian Density by executing a Legendre transformation with respect to the dynamical variables $\frac{\partial \Phi}{\partial t}$ and $\frac{\partial A}{\partial t}$. We arrive at:

$$\mathcal{H} = \Pi_\Phi \dot{\Phi} + \Pi_A \dot{A} - \mathcal{L} = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

Which is (somewhat unsurprising) energy density of the electromagnetic field, which we could have guessed. But, we would have not gotten the definition of the canonical momenta and positions from just guessing the Hamiltonian density, this is however an important ingredient in the quantization process, as they are crucial in the definition of observables to the system.

2.2 Field Quantization in Space

Now we quantize the electromagnetic field by adding a "hat" to all fields. I.e. we promote them from scalar (or vectorial) fields to operator fields. By construction these obey certain commutation relations:

$$[\hat{A}^i(r, t), \hat{A}^k(r', t')] = 0$$
$$[\hat{\Pi}_A^i(r, t), \hat{\Pi}_A^k(r', t')] = 0$$

We will later see that these have a meaning beyond pure mathematics: it means, that one can make independent measurements of the quantum field and independent measurements of its momentum (i.e. the $E$ and $B$ field) at two different points in space at arbitrary order, without mutual influence of the result.

As opposed to the classical theory, we however also get the following mixed commutation relation

$$[\hat{A}_i(r, t), \hat{\Pi}_A^j(r', t')] = i\hbar \Delta_{ij}(r - r')$$

$$\Delta_{ij}(r) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}r} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right)$$

Where the second equations comes into play, due to the relativistic nature of the fields and the third term is basically an ordinary $\delta$-function, which is corrected for the divergence-free nature of the EM-field (i.e. that we have only two-polarizations for three spatial degrees of freedom). $\mathbf{k}$ is the wave-vector of a plane wave, which we will introduce shortly.

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This means that we cannot measure the field and its momentum independently at the same point in space and time. If you measure both, its result will depend on the order of the measurement.

2.3 Introduction of Quantum Plane Waves Modes

We can, of course decompose each quantum field into quantum plane waves. We will later see that due to the construction of the quantum fields, these quantum plane waves are eigenstates of the Hamiltonian-Operator of the system and thus remain shape invariant (expect for a phase term) under the evolution of time. They also have, due to the construction, the same spatial form and dispersion relation.

\[
\hat{A}(r, t) = \sum_{\lambda} \int \frac{dk}{(2\pi)^3} \frac{\hbar}{\epsilon_0} \left\{ \epsilon_{\lambda j}(k) \hat{a}_\lambda(k) u(k; r, t) + c. c. \right\}
\]

\[
u(k; r, t) = \frac{e^{i(kr - \omega_k t)}}{\sqrt{(2\pi)^32\omega_k}}
\]

This means

• The only thing which is changed in QED is that the field strength in each plane wave mode is now a not a scalar number but an operator in itself
• Plane wave are still invariant modes of the system.
• The nature of polarization does not change

As we will later mostly just look into plane waves, it now makes sense to derive commutation relations for their operators. These are

\[
[\hat{a}_\lambda(k), \hat{a}^\dagger_{\lambda'}(k')] = \delta_{\lambda\lambda'}\delta(k - k')
\]

\[
[\hat{a}_\lambda(k), \hat{a}_{\lambda'}(k')] = [\hat{a}^\dagger_\lambda(k), \hat{a}^\dagger_{\lambda'}(k')] = 0
\]

Which means that the state of any plane wave can be determined independently from the state of any other plane wave, expect for the state of a plane wave and its conjugate. To move ahead somewhat: you cannot determine the field and its derivative aka. The electric and the magnetic field of one mode (think about measuring currents and voltages in Electronics --> both measurements necessarily influence each other!!!).

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In order to derive these equations we have made use of the scalar product, which is needed to decompose the state of the quantum field into the appropriate modes. This modal scalar product is defined as:

\[
(\phi, \psi) = i \int d\mathbf{r}(\phi^* \partial_t \psi - (\partial_t \phi)^* \psi)
\]

This scalar product has some important features:

• It is a mathematical representation of the continuity equation and thus time-independent
• It defines a complete and orthogonal set of modes (this will not be derivied here).

Using this scalar product we can invert the equation between \(\hat{A}\) and \(\hat{a}_j\), namely (see seminar for complete derivation)

\[
\hat{a}_\lambda'(k') = \frac{\epsilon_0}{\hbar} \sqrt{u(k; r, t)\epsilon_\lambda', \hat{A}(r, t)}
\]

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This equation basically gives us a recipe on how we can decompose any field into plane wave modes. Note that we have left out the +c.c for notational brevity; these however just end up as +0 in the integrals. In this derivation we have also implicitly proven, that the modes are orthogonal with respect to each other and that they are complete.

2.4 Non Plane-Wave-Modes

Of course, plane waves are by no means the only set of modes that can be used to describe the field. As they are, however, complete we can construct any basis set \( v(\kappa, r, t) \) of plane waves from a superposition of them, i.e.

\[
v(\kappa; r, t) = \int d\kappa V(\kappa, \kappa)u(\kappa; r, t)
\]

Note that \( \kappa \) is now any set of indices, which enumerates the new basis set and \( V(\kappa, \kappa) \) is a unitary matrix, i.e. \( V^* = V^{-1} \). As with the plane wave modes above they share the same relationship with the operator field \( \hat{A} \) as the plane wave operators:

\[
\hat{B}_\lambda' = \sqrt{\frac{\varepsilon_0}{\hbar}}(v(\kappa)\epsilon_\lambda', \hat{A})
\]

This relation can also be evaluated in terms of the plane wave operators \( \hat{a}_\lambda(k) \), this yields:

\[
\hat{B}_\lambda' = \sum_\lambda \int dk [\epsilon_\lambda'(\kappa)\epsilon_\lambda(k)(v(\kappa), u(k))\hat{a}_\lambda(k) + \epsilon_\lambda^*(\kappa)\epsilon_\lambda^*(k)(v(\kappa), u^*(k))\hat{a}_\lambda^*(k)]
\]

Note that by the construction the modes \( \hat{B}_\lambda' \) have the same commutation relation as \( \hat{a}_\lambda(k) \). Also note that we have been somewhat sloppy in the transformation of the polarization eigenmodes, but these can be included in the equations in a straightforward manner.

It is indeed useful as plane waves are kind of non-physical. They have unbounded energy and they are of infinite extent. Particularity for experiments it often makes sense to describe the filed in something more...close to reality.

2.4.1 Example 1: Gaussian Modes

For the introduction of Gaussian modes we will first neglect polarization altogether (assuming, i.e. Linear polarization, which is in itself an approximation, as the different plane waves that compose a Gaussian have different polarization states) and then we assume paraxiality, i.e. The beam diameter is much larger than the wavelength of light. We will also assume that the field is harmonic with a frequency of \( \omega_0 \) (i.e. It is of a single frequency) and that its propagation direction is mostly along the \( z \)-axis. Thus

\[
k_z \approx k \left( 1 - \frac{k_x^2 + k_y^2}{k^2} \right) \quad k = \frac{\omega}{c}
\]

Under these assumptions, the Gaussian Mode takes the form

\[
\hat{A}_0(r, t) = \int d\omega (\hat{A}(\omega)A_0(r, \omega)e^{i\omega t} + c.c)
\]

\[
A_0(r, \omega) = \frac{4\pi}{\sqrt{\epsilon_0 s^2(z)}} \exp \left( i k_z - \frac{x^2 + y^2}{s^2(z)} \right)
\]

\[
s^2(z) = w_0^2 + \frac{2iz}{k}
\]
\[ z_R = \frac{\pi w_0^2}{\lambda} \quad w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \quad NA = \frac{w_0}{z_R} = \frac{\lambda}{\pi w_0} \]

One can indeed show, that these transverse modal fields are minimum uncertainty localized transverse modes, i.e.

\[ \Delta k \Delta x = \frac{1}{4\pi} \int dk |f(k)|^2 \to w_0 NA = \frac{\lambda}{\pi} \]

Among other things this means, that Gaussian modes have the least divergence of all beams for a given cross section. They are therefore well suited for long-range communication, as they require the smallest telescopes.

2.4.1 Example 1: Gauss-Laguerre Modes

We can extend on some of the relations found for the Gaussian Modes and extend these onto a complete set of Eigenmodes with rotational symmetry, i.e. we introduce

\[ x = r \cos \varphi \quad y = r \sin \varphi \quad \sigma = x + iy \]

In Plane wave representation the Gauss-Laguerre modes have the form

\[ A_{lm}(r, \omega) = \frac{4\pi (-1)^{l+m} l!}{\sqrt{\varepsilon_0 s^2(l+m+1)(z)}} e^{im\varphi} L_l^m \left( \frac{r^2}{s^2(z)} \right) e^{ikz - \frac{r^2}{2s(z)}} + c. c \]

\[ u_{lm}^{LG}(k, r, t) = A_{lm}(r, \omega) e^{i\omega t} \]

\[ \Rightarrow \hat{A}(r, t) = \sqrt{\frac{\varepsilon_0}{\hbar}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int dk \hat{a}_{lm}(k) u_{lm}^{LG}(k, r, t) + c. c \]

Note that \( s(z) \) was defined in the last chapter. Also note that the \( \varphi \)-dependency is exclusively a phase term \( \sim \exp(-im\varphi) \). Thus one can easily see that these modes are eigenfunctions to the operator, which measures the \( z \)-coordinate of the angular impulse \( \hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial \varphi} \) with the Eigenvalue \( \hbar m \).

\[ \hat{L}_z u_{lm}^{LG}(k, r, t) = m \hbar u_{lm}^{LG}(k, r, t) \]

These beams thus carry a quantized and measurable orbital angular momentum. As this is a discrete quantity it can be used conveniently to transport information with more than a bit per photon (i.e. A so-called qdit). Also note that this information transfer is quite robust: the angular momentum is a compatible measurable to both the direction \( (k_0) \) of the beam, its frequency \( (\omega) \) as well as its overall...
impulse ($l$). Propagation through air typically induced perturbation along $k_0$ and $l$ but very little on $m$. Information encoded in these modes is thus also robust.

2.5 Polarization modes
So far we have pretty much ignored the polarization aspects of the modes. We shall now have a closer look at these. As with classical EM-theory these can be represented with Jones Vectors

$$\epsilon(k) = \begin{bmatrix} \epsilon_1(k) \\ \epsilon_2(k) \end{bmatrix}$$

Where we have assumed, without loss of generality, that $k = k_z e_z$. Then we can find a few single basis-vector systems, in which we can describe the polarization state of light:

- linear HV: $\epsilon = h \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} = h|h\rangle + v|v\rangle$
- linear diagonal: $\epsilon = u \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + d \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = u|u\rangle + d|d\rangle$
- linear: $\epsilon = l_1 \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix} + l_2 \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix} = l_1|l_1\varphi\rangle + l_2|l_2\varphi\rangle$
- circular: $\epsilon = l \begin{bmatrix} 1 \\ i \end{bmatrix} + r \begin{bmatrix} 1 \\ -i \end{bmatrix} = l|l\rangle + r|r\rangle$

2.6 Quantization of the Hamiltonian
Now that we have introduced the quantum modal operators $\hat{a}_\lambda(k)$ it is quite straightforward to show that the QET analogon of the Hamilton-Opertaor is (Derivation in Seminar)

$$\hat{H} = \sum_\lambda \int dk \frac{\hbar \omega(k)}{2} \left( \hat{a}_\lambda^\dagger(k) \hat{a}_\lambda(k) + \hat{a}_\lambda^\dagger(k) \hat{a}_\lambda(k) \right)$$

The Hamiltonian has the following commutation relations (derivation in Seminar)

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Lecture in Quantum Communication, WS 2018/2019, Friedrich-Schiller-University, Jena
Fabian Steinlechner and Falk Eilenberger

\[
\begin{align*}
[\hat{H}, \hat{a}_\lambda(k)] &= -\hbar \omega \hat{a}_\lambda(k) \\
[\hat{H}, \hat{a}_\lambda^\dagger(k)] &= \hbar \omega \hat{a}_\lambda^\dagger(k)
\end{align*}
\]

Let’s assume that we have found eigenstates for the Hamiltonian, such that

\[
\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle
\]

Then these can be combined with the operator commutation equation

\[
\begin{align*}
\hat{H} \hat{a}_\lambda(k) |\psi_n\rangle &= \hat{a}_\lambda(k) \hat{H} |\psi_n\rangle - \hbar \omega \hat{a}_\lambda(k) |\psi_n\rangle \\
&= (E_n - \hbar \omega) \hat{a}_\lambda(k) |\psi_n\rangle \\
\hat{H} \hat{a}_\lambda^\dagger(k) |\psi_n\rangle &= (E_n + \hbar \omega) \hat{a}_\lambda^\dagger(k) |\psi_n\rangle
\end{align*}
\]

This means that \(\hat{a}_\lambda(k)\) reduces the eigenvalue (i.e. energy) of the state \(|\psi_n\rangle\) by a certain quantity but it’s still an eigenstate. Same for \(\hat{a}_\lambda^\dagger(k)\) just the it reduced the eigenvalue (i.e. energy). As the Eigenvalues of \(\hat{H}\) must be bound from below (it’s a energy after all and negative energy is kind of hard to come by!) there must be a ground state for which

\[
\hat{H} |\psi_0\rangle = 0 \quad \forall \lambda, k
\]

This is called the quantum-vacuum state and ist often also denoted as \(|0\rangle\). Let’s calculate ist energy

\[
\begin{align*}
\hat{H} |\psi_0\rangle &= \sum_{k, \lambda} \int dk \hbar \omega (k) \hat{a}_\lambda^\dagger(k) \hat{a}_\lambda(k) |\psi_0\rangle \\
&= 2 \int dk \delta(k - k') |\psi_0\rangle
\end{align*}
\]

This term is the quantum vacuum energy \(\mathcal{E}\). It diverges and must be removed for all practical calculation of the energy. It’s however not unphysical. It lead e.g. to the Lamb-Shift, the Casimir-Force, and the Quantum-Unruh-Effect (dynamical Casimir Effect).

The divergence does however occur for two reasons. (1) because of the infinite sum and (2) because of the \(\delta\)-function. It turns out the latter is a problem of the temporal structure of the modes, which we have ignored so far, i.e. we have assumed them to be ill-behaved, infinite, naughty harmonics. But we can fix this. However, we first introduce (plane-wave) Fock-States \(|n_{k, \lambda}\rangle\) for every mode denoted by \(k\) and \(\lambda\) by repeatedly applying \(\hat{a}_\lambda^\dagger(k)\) \(n\) times to \(|\psi_0\rangle\). These then have the relative energy:

\[
E_n(k) = \hbar \omega(k)n
\]

We also define the number-operator

\[
\hat{n} = \sum_{k, \lambda} \int dk \hat{a}_\lambda^\dagger(k) \hat{a}_\lambda(k)
\]

Which has a well-defined meaning for Fock-States \(|n_{k, \lambda}\rangle\)

\[
\hat{n} |n_{k, \lambda}\rangle = n |n_{k, \lambda}\rangle
\]

As \(\hat{a}_\lambda^\dagger(k)\) and \(\hat{a}_\lambda(k)\) can be used to move us up and down the ladder of Fock-States, we call then ladder-operators or creation and annihilation operators.

The plane-wave Fock-States still have the above-mentioned naughtiness, i.e.

\[
\begin{align*}
\langle 1_{k', \lambda'} | 1_{k'\lambda} \rangle &= \langle 0_{k, \lambda} | \hat{a}_{\lambda'}^\dagger(k) \hat{a}_{\lambda'}^\dagger(k) |0_{k'\lambda} \rangle \\
&= \langle 0_{k, \lambda} | \hat{a}_{\lambda'}^\dagger(k) \hat{a}_{\lambda'}(k) |0_{k'\lambda} \rangle + \delta_{ij} \delta(k - k') \\
&= \delta_{ij} \delta(k - k')
\end{align*}
\]

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We'll now simply make a basis transformation into a set of modes $f_{j,\lambda}$, which are centered around a wave-vector $k_j$ and which themselves form an orthonormal basis. We can then decompose the quantum field $\hat{A}$ into these modes

$$\hat{b}_{j\lambda} = \sqrt{\frac{\hbar}{2\pi}} (\varepsilon_{\lambda j} f_{\lambda j} \hat{A})$$

$$= \int dk \left( \alpha_j(k) \hat{a}_j(k) + \beta_j(k) \hat{a}^+_j(k) \right)$$

For the sake of simplicity we can assume that $\beta_j(k) = 0$ and of course we know $\int dk |\alpha_j(k)|^2 = 1$. By construction the operators, fulfill the commutation relations

$$[\hat{b}_{j\lambda}, \hat{b}^\dagger_{j'\lambda'}] = \delta_{j,j'} \delta_{\lambda,\lambda'}$$
$$\hat{b}_{j\lambda}, \hat{b}^\dagger_{j'\lambda'}] = [\hat{b}^\dagger_{j\lambda}, \hat{b}^\dagger_{j'\lambda'}] = 0$$

We can then now attempt to construct a set of Fock Modes for this basis starting from the first mode

$$|1_{j\lambda}\rangle = \hat{b}^\dagger_{j\lambda} |0\rangle$$

$$= \int dk \alpha^*_j(k) \hat{a}^+_j(k)|0\rangle$$

This, however, does fail as this is not normalizable

$$\langle 1_{j\lambda}|1_{j'\lambda'}\rangle = \int dk dk' \alpha_j(k) \alpha^*_j(k') \langle 0|\hat{a}_j(k)\hat{a}^+_j(k')|0\rangle$$

$$= \int dk dk' \alpha_j(k) \alpha^*_j(k') \langle 0|\hat{a}^+_j(k')\hat{a}_j(k) + \delta_{\lambda,\lambda'} \delta(k-k')|0\rangle$$

$$= \delta_{\lambda,\lambda'} \int dk dk' \alpha_j(k) \alpha^*_j(k') \delta(k-k')(0|0)$$

$$= \delta_{\lambda,\lambda'} \int dk \alpha_j(k) \alpha^*_j(k)$$

$$= \delta_{\lambda,\lambda'}$$

This is now well-behaved! Yay. Keep in mind that the function $\alpha_j(k)$ may be very localized, such that from an experimental point of view here is very little difference to a plane wave here. We'll therefore in the future often forget the difference between $\hat{b}_{j\lambda}$ and $\hat{a}_j(k)$.

Let us now used the normalized mode operators to properly normalize their respective Fock Modes, which we could not do previously:

$$|n_{j\lambda}\rangle = \frac{1}{\sqrt{n!}} (\hat{b}^\dagger_{j\lambda})^n |0\rangle$$

$$\hat{b}^\dagger_{j\lambda} |n_{j\lambda}\rangle = \sqrt{n} |n-1_{j\lambda}\rangle$$

$$\hat{b}_{j\lambda} |n_{j\lambda}\rangle = \sqrt{n+1} |n+1_{j\lambda}\rangle$$

$$\hat{n}_{j\lambda} |n_{j\lambda}\rangle = \hat{b}^\dagger_{j\lambda} \hat{b}_{j\lambda} |n_{j\lambda}\rangle = n |n_{j\lambda}\rangle$$

The complete Hamiltonian can then be written as

$$\hat{H} = \sum \lambda \int df \hbar \omega_\lambda \hat{n}_{j\lambda}$$

This has a quite straightforward interpretation

- Each mode $f_{j,\lambda}$ is filled with $n$ particles, if it is in state $|n_{j\lambda}\rangle$
- The $\hat{b}^\dagger_{j,\lambda}$ operator creates one particle in mode $f_{j,\lambda}$
- The $\hat{b}_{j,\lambda}$ operator destroys a particle in mode $f_{j,\lambda}$
- Each of these particles is called a photon.
- Each photon has a set of observables.