Imaging and Aberration Theory

Lecture 3: Eikonal
2017-10-30
Herbert Gross
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<td>Vectorial aberrations, generalized surface contributions, Aldis theorem, intrinsic and induced aberrations, revertability</td>
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1. Fermat principle
2. Principle of stationary phase
3. Hamiltonian approach
4. Eikonal
5. Refracting surface
6. Perfect imaging
7. Ray-wave relations
8. Approximation of geometrical optics
9. Raytrace in inhomogeneous media
- Fermat principle: the light takes the ray path, which corresponds to the shortest time of arrival.

- The realized path is a minimum and therefore the first derivatives vanish:
  \[ \delta L = \delta \int_{P_1}^{P_2} n(x, y, z) \, ds = 0 \]
  here \( s \) is the arc length along the ray path.

- Several realized ray paths have the same optical path length:
  \[ L = \int_{P_1}^{P_2} n \cdot \vec{s} \cdot d\vec{r} = \text{const.} \]

- The principle is valid for smooth and discrete index distributions.
The Fermat principle states, that a modified ray path must have a larger optical path length.
The realized path is a minimum and therefore the first derivatives to path variables must vanish:

\[
\frac{\partial L(u, v)}{\partial u} = 0, \quad \frac{\partial L(u, v)}{\partial v} = 0
\]

\(u\) and \(v\) are arbitrary coordinates on the surface, which indicate the ray intersection point.

The phase has a stationary point with minimal optical path length.
In the special case of conjugated points A and B, this is true for every ray in a perfect system.
Principle of Stationary Phase

- Principle of stationary phase from an illustrative point of view
- Oscillatory parts are cancelled out
- The light delivers constructive interference in those directions, which has stationary phase contributions: the ray direction perpendicular to the wave
- Critical are stationary phase contributions of boundary points
Principle of Stationary Phase

- Principle of stationary phase from wave optical viewpoint

- Diffraction integral as plane wave decomposition with phase and slowly varying amplitude

\[
E(x, y, z) = \int \int E(s_x, s_y) \cdot e^{\frac{2\pi i}{\lambda} (xs_x + ys_y + zs_z)} ds_x ds_y
\]

\[
E(x, y, z) = \int \int A(s_x, s_y) \cdot e^{\frac{2\pi i}{\lambda} L(s_x, s_y)} \cdot e^{\frac{2\pi i}{\lambda} [xs_x + ys_y + zs_z]} ds_x ds_y
\]

- Oscillatory contributions cancel out except the point of stationary phase

\[
\frac{d}{ds_x} [L(s_x, s_y) + xs_x + ys_y + zs_z] = 0 \quad , \quad \frac{d}{ds_y} [L(s_x, s_y) + xs_x + ys_y + zs_z] = 0
\]

- This gives the solutions

\[
x = -\frac{dL}{ds_x} + \frac{s_x}{s_z} \cdot z \quad , \quad y = -\frac{dL}{ds_y} + \frac{s_y}{s_z} \cdot z
\]

with the normalization relation

\[
s_x^2 + s_y^2 + s_z^2 = 1
\]

- The light propagates along the ray path perpendicular to the phase / wave front
- Rays and waves carry the same information
- Wave surface is perpendicular on the rays
- Wave is purely geometrical and has no diffraction properties
Law of Malus-Dupin

- Law of Malus-Dupin:
  - equivalence of rays and wavefronts
  - both are orthonormal
  - identical information

- Condition:
  No caustic of rays

- Mathematical:
  Rotation of Eikonal vanish
  \[ \text{rot}(n \cdot \vec{s}) = 0 \]

- Optical system:
  Rays and spherical waves orthonormal
The point eikonal or optical path length serves as the Lagrange function written in coordinate representation with derivative with respect to \( z \) gives the optical Lagrange formulation

\[
L = \int n(\vec{r}) \, d\vec{r}
\]

\[
ds = |d\vec{r}| = \sqrt{dx^2 + dy^2 + dz^2}
\]

\[
dz \cdot \sqrt{1 + \left( \frac{dx}{dz} \right)^2 + \left( \frac{dy}{dz} \right)^2} = dz \cdot \sqrt{1 + \dot{x}^2 + \dot{y}^2}
\]

\[
L(x, y, \dot{x}, \dot{y}, z) = n(x, y, z) \cdot \sqrt{1 + \dot{x}^2 + \dot{y}^2}
\]

With the definition of the impulse variables

\[
p_x = \frac{\partial L}{\partial \dot{x}} = n \cdot \frac{\dot{x}}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}} = n \cdot \frac{dx}{ds} = n \cdot s_x
\]

\[
p_y = \frac{\partial L}{\partial \dot{y}} = n \cdot \frac{\dot{y}}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}} = n \cdot \frac{dy}{ds} = n \cdot s_y
\]

The equation of motion reads in vectorial notation corresponds to the eikonal equation

\[
\frac{d}{ds} \left( n \cdot \frac{d\vec{r}}{ds} \right) = \nabla n \quad \quad \quad n(\vec{r}) = |\nabla L|
\]

The Hamilton version is given by the Legendre transform

\[
H(x, y, p_x, p_y) = p_x \cdot x' + p_y \cdot y' - L
\]

\[
= -\sqrt{n^2 - p_x^2 - p_y^2}
\]
Eikonal Formulation of Imaging

- A ray is described in a plane of constant z by 4 variables
  - point x,y
  - direction or angles $s_x, s_y$

- An optical system transmits an initial ray into a ray in the image space

- In the most general mathematical description, 4 functions control this ray transform

$$
x' = L_1(x, y, s_x, s_y), \quad y' = L_2(x, y, s_x, s_y)\\
{s_x}' = L_3(x, y, s_x, s_y), \quad {s_y}' = L_4(x, y, s_x, s_y)$$

- The Fermat principle restricts this most general approach to one single function, the so called eikonal function L

$$\left( x', y', s'_x, s'_y \right) = L(x, y, s_x, s_y)$$

- In reality, there are only 4 degrees of freedom:
  for a pre-given initial ray, the transferred ray is fixed
Eikonal Formulation of Imaging

- There are 4 possible options to formulate the problem concerning the choice of the independent variables combining object and image space

<table>
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<th>image space</th>
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<td>point</td>
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<tr>
<td>x,y</td>
<td>x',y'</td>
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<tr>
<td>s_x,s_y</td>
<td>s'_x,s'_y</td>
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<tr>
<td>x,y</td>
<td>s'_x,s'_y</td>
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<td>x',y'</td>
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- The different eikonal functions can be calculated via a variable transform by a Legendre transform
- The eikonal functions have singularities, which limits their application
  Example: if conjugated planes are considered, x',y' are fully determined by x,y. Therefore there are no 4 independent variables and the point eikonal fails
- There is not one description, which is valid and useful for all possible cases
Point Eikonal

- Point Eikonal \( L(x,y,x',y') \):
  Optical path length from point \( P(x,y) \) in object space to point \( P'(x',y') \) in image space

- Total differential of the Hamilton Eikonal for purely transverse directions and \( z = z' = 0 \)
  \[
  dL_p(x, y, x', y') = n'(s'_x dx' + s'_y dy') - n(s_x dx + s_y dy)
  \]

- Differential equations
  \[
  \frac{\partial L_p}{\partial x} = -ns_x, \quad \frac{\partial L_p}{\partial y} = -ns_y
  \]
  \[
  \frac{\partial L_p}{\partial x'} = n's'_x, \quad \frac{\partial L_p}{\partial y'} = n's'_y
  \]

- Physical interpretation:
  Given point in object and image space:
  Integration of the equations give the corresponding directions

- Not valid in conjugated planes:
  \( x', y' \) does not depend on path \( s, s' \)
Point-Angle Eikonal

- Legendre-transformation of point eikonal:
  point angle eikonal
  \[ L_{PA}(x, y, s_x', s_y') = L_p(x, y, x', y') - n'(x's_x' + y's_y') \]

- Total differential
  \[ dL_{PA}(x, y, s_x', s_y') = -n'(x'ds_x' + y'ds_y') - n'(s_xdx + s_ydy) \]

- Derivatives:
  \[ \frac{\partial L_{PA}}{\partial x} = -ns_x \quad \frac{\partial L_{PA}}{\partial y} = -ns_y \quad \frac{\partial L_{PA}}{\partial s_x'} = -n'x' \quad \frac{\partial L_{PA}}{\partial s_y'} = -n'y' \]

- \( L_{PA} \) defines a ray with starting point and final direction

- The point angle eikonal can be used for conjugated planes

- Point angle eikonal is not applicable for an afocal system:
  \( s_x', s_y' \) are independent of \( x, y \)
Point-Angle Eikonal

- Interpretation of the point angle eikonal
- Optical path length from the starting point P to the point Q' in the image space, which corresponds to the perpendicular projection

\[ \mathbf{s}'_x = n \sin(u) \]
Angle Eikonal

- Corresponding Legendre transform gives the pure angle eikonal according to Schwarzschild

\[ dL_A(s_x, s_y, s'_x, s'_y) = -n'(x' ds'_x + y' ds'_y) + n \cdot (x ds_x + y ds_y) \]

- Vectorial formulation

\[ dL_A(\bar{s}, \bar{s}') = n \cdot (\bar{r} - \bar{a}) \cdot d\bar{s} - n' \cdot (\bar{r}' - \bar{a}') \cdot d\bar{s}' \]

- The ray is defined by its directions only

- Differential equations

\[ \frac{\partial L_A}{\partial s_x} = nx \quad \frac{\partial L_A}{\partial s_y} = ny \quad \frac{\partial L_A}{\partial s'_x} = -n'x' \quad \frac{\partial L_A}{\partial s'_y} = -n'y' \]

- Interpretation:
  optical path length between the feet points Q, Q' perpendicular to the ray
Reciprocity Relation

- Point angle eikonal
  \[ \frac{\partial L_{AP}}{\partial s_x} = nx \quad \frac{\partial L_{AP}}{\partial x'} = n' s'_x \]

- Independence of mixed derivatives
  \[ \frac{\partial^2 L_{AP}}{\partial s_x \partial x} = n \quad \frac{\partial^2 L_{AP}}{\partial x' \partial s'_x} = n' \]

- Result:
  - reciprocity relation
    \[ n \cdot \partial s_x \partial x = n' \cdot \partial s'_x \partial x' \]
  - relation between lateral and angle magnification
    \[ n \cdot m_A = n' \cdot m \]
Optical Path Length of an Optical System

- A ray from P goes through a system to P'
- Change of initial point P to Q
- Comparison of optical path of P and Q via A: Difference: Hamilton eikonal

\[ \delta L = \overline{QQ'} - \overline{PP'} = \overline{QAA'}\overline{Q'} = \overline{PAA'}\overline{P'} = \overline{P'R'} - \overline{PR} \]

\[ = n'dr' \cos \theta' - ndr \cos \theta \]

\[ = n's' \cdot dr' - n\bar{s} \cdot d\bar{r} \]

- Total differential: spatial point eikonal
- Differential equations

\[ \frac{\partial L_{Ham}}{\partial x} = -ns_x \]
\[ \frac{\partial L_{Ham}}{\partial y} = -ns_y \]
\[ \frac{\partial L_{Ham}}{\partial z} = -ns_z \]

\[ \frac{\partial L_{Ham}'}{\partial x'} = n's_x' \]
\[ \frac{\partial L_{Ham}'}{\partial y'} = n's_y' \]
\[ \frac{\partial L_{Ham}'}{\partial z'} = n's_z' \]

- Change of initial point: the change in the final point is fixed by the eikonal
Paraxial Point Eikonal for a Refracting Surface

- Refracted ray by a spherical dielectric interface path difference

- Paraxial approximation:
  Taylor expansion for small $x$, $x'$, $r$, $z$

- Stationary phase condition

$$L(r) = n \cdot \sqrt{(z - s)^2 + (r - x)^2} + n' \cdot \sqrt{(-z + s')^2 + (r - x')^2}$$

$$L(r) = \left[ -ns + n's' - \frac{nx^2}{2s} + \frac{n'x'^2}{2s'} \right] + r \cdot \left[ + \frac{nx}{s} - \frac{n'x'}{s'} \right] + r^2 \cdot \frac{1}{2} \left[ - \frac{n}{s} + \frac{n'}{s'} - \frac{n' - n}{R} \right]$$

$$= A + Br + Cr^2$$

- Angles $u$, $u'$

$$u = \frac{r - x}{-s}, \quad u' = \frac{r - x'}{s'}$$
Paraxial Point Eikonal for a Refracting Surface

- Paraxial imaging conditions for $B = 0$ and $C = 0$

1. Lens makers formula

$$\frac{nx}{s} - \frac{n'x'}{s'} = 0, \quad -\frac{n}{s} + \frac{n'}{s'} - \frac{n'-n}{R} = 0$$

2. definition of magnification

$$m = \frac{x'}{x} = \frac{n \cdot s'}{n' \cdot s}$$
The point angle eikonal of a lens is suitable to describe the aberrations.

The derivatives give the deviations:
1. of the position:
   transverse aberrations as deviation from perfect paraxial location

2. of the direction:
   angle aberrations correspondingly

The mixed derivative of the eikonal gives a relationship between the transverse aberration components.
Angle Eikonal for a Refracting Surface

- P point on ray,  A arbitrary point on axis

- Real surface
  \[ z = \frac{x^2 + y^2}{2R} + \frac{(x^2 + y^2)^2}{8R^3} \cdot (1+b) + \ldots \]

- Angle eikonal
  \[ dL_A(\vec{s}, \vec{s}') = n \cdot (\vec{r} - \vec{a}) \cdot d\vec{s} - n' \cdot (\vec{r}' - \vec{a}') \cdot d\vec{s}' \]

- In coordinate representation
  \[ dL_A = n \cdot \left[ x \cdot ds_x + y \cdot ds_y + (z-a) \cdot ds_z \right] - n' \cdot \left[ x' \cdot ds'_x + y' \cdot ds'_y + (z-a') \cdot ds'_z \right] \]
  \[ s_z = \sqrt{1 - s_x^2 - s_y^2}, \quad s'_z = \sqrt{1 - s'_x^2 - s'_y^2} \]
Angle Eikonal for a Refracting Surface

- Result for 4th order Taylor approximation (x, y, are still parameters, which should be eliminated)

\[ dL_A = -n \cdot a + n' \cdot a' \]

\[ + n \cdot \left[ x \cdot s_x + y \cdot s_y + \frac{x^2 + y^2}{2R} + \frac{a}{2} \cdot (s_x^2 + s_y^2) \right] \]

\[ - n' \left[ x \cdot s'_x + y \cdot s'_y + \frac{x^2 + y^2}{2R} + \frac{a'}{2} \cdot (s'_x^2 + s'_y^2) \right] \]

\[ + n \cdot \left[ \frac{1+b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s_x^2 + s_y^2)}{4R} + \frac{a}{8} \cdot (s_x^2 + s_y^2)^2 \right] \]

\[ - n' \left[ \frac{1+b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s'_x^2 + s'_y^2)}{4R} + \frac{a'}{8} \cdot (s'_x^2 + s'_y^2)^2 \right] \]

- Use of refraction law to eliminate x, y

\[ dL_A^{(4)} = S_1 \cdot u^2 + S_2 \cdot v^2 + S_3 \cdot w^2 + S_4 \cdot u \cdot v + S_5 \cdot u \cdot w + S_6 \cdot v \cdot w \]

with the rotational invariants

\[ u = s_x^2 + s_y^2 , \quad v = s'_x^2 + s'_y^2 , \quad w = s_x s'_x + s_y s'_y \]

and coefficients \( S_j \)
Angle Eikonal for a Refracting Surface

- Coefficients $S_j$

Description of the optical path as a function of
1. system data: $n$, $n'$, $R$, $b$
2. ray parameter $a$, $a'$

\[
S_1 = \frac{a}{8n^3} - \frac{R}{4(n-n')^2} \cdot \left[ \frac{1}{n} + \frac{1+b}{2(n'-n)} \right]
\]

\[
S_2 = - \frac{a'}{8n'^3} - \frac{R}{4(n-n')^2} \cdot \left[ -\frac{1}{n'} + \frac{1+b}{2(n'-n)} \right]
\]

\[
S_3 = - \frac{(1+b)R}{2(n'-n)^3}
\]

\[
S_4 = - \frac{R}{4(n-n')^2} \cdot \left[ \frac{1}{n} - \frac{1}{n'} + \frac{1+b}{n'-n} \right]
\]

\[
S_5 = \frac{R}{2(n-n')^2} \cdot \left[ \frac{1}{n} + \frac{1+b}{n'-n} \right]
\]

\[
S_6 = - \frac{R}{2(n-n')^2} \cdot \left[ -\frac{1}{n} + \frac{1+b}{n'-n} \right]
\]
Perfect Imaging

- **Eikonal theory:**
  - perturbation method
  - zero order is the paraxial approximation
  - higher order perturbation corresponds to aberrations

- **Perfect imaging as special cases:**
  1. Stigmatic imaging with conic sections finite-finite (ellipsoid), infinite-fine (parabola)
  2. Special refractive index distributions, e.g. Maxwellian fish-eye
  3. For infinitesimal field size and finite aperture cone: aplanatic imaging condition

- **Imperfect lens: magnification and transverse aberrations**
  \[ x' = m \cdot x + \Delta x, \quad y' = m \cdot y + \Delta y \]

- **From point angle eikonal differential equations:**
  \[ x' = -\frac{1}{n'} \frac{\partial L_{PA}(x, y, s'_x, s'_y)}{\partial s'_x}, \quad y' = -\frac{1}{n'} \frac{\partial L_{PA}(x, y, s'_x, s'_y)}{\partial s'_y} \]
Perfect Imaging

- Perfect imaging as special cases:
  
  Examples

- Maxwell lens

  \[ n(x, y, z) = \frac{2n_{env}}{1 + \frac{x^2 + y^2 + z^2}{R^2}} \]

- Mikhaelian lens

  \[ n(r) = \frac{n_0}{\cosh\left(\frac{\pi}{2d} \cdot r\right)} \]
What is ‘Ideal‘?

- The notation ‘ideal‘ imaging is not unique
- Ideal is in any case the location of the image point
- The geometrical ray paths can be different for
  1. paraxial
  2. ideal / linear collineation
  3. aplanatic
- The photometric properties are different due to non-equidistant sampling
- If a perfect lens is idealized in a software as one surface, there are principal discrepancies in the location of the intersection points
Abbe Sine Condition

- If for example a small field area and a widespread ray bundle is considered, a perfect imaging is possible.

- The eikonal with the expression

\[ \delta L = n' \tilde{s}' \cdot d\tilde{r}' - n\tilde{s} \cdot d\tilde{r} \]

can be written for \( \delta L = 0 \) as

\[ n \cdot \tilde{s} \cdot d\tilde{r} = n' \cdot \tilde{s}' \cdot d\tilde{r}' \]

\[ n \cdot dr \cdot \cos \theta = n' \cdot dr' \cdot \cos \theta' \]

\[ n \cdot \cos \theta = n' \cdot m \cdot \cos \theta' \]

- In the special case of an angle 90° we get with \( \cos(\theta) = \sin(u) \) the Abbe sine condition

\[ m = \frac{n \sin u}{n' \sin u'} = \frac{y'}{y} \]

with the lateral magnification

\[ m = \frac{d\tilde{r}'}{dr} \]
Ray-Wave Relationships

- Concrete calculation of wave aberration: addition of discrete optical path lengths (OPL)
- Reference on chief ray and reference sphere (optical path difference)
- Relation to transverse aberrations
- Conversion between longitudinal transverse and wave aberrations
- Scaling of the phase / wave aberration:
  1. Phase angle in radiant
  2. Light path (OPL) in mm
  3. Light path scaled in \( \lambda \)

\[
l_{OPL} = \int_{OE}^{AP} n \cdot d\vec{r}
\]

\[
\Delta_{OPD}(x, y) = l_{OPL}(x, y) - l_{OPL}(0, 0)
\]

\[
\frac{\partial W}{\partial y_p} = - \frac{\Delta y'}{R - W} \approx - \frac{\Delta y'}{R}
\]

\[
\Delta s' = \frac{R}{y_p} \cdot \Delta y' = \frac{\Delta y'}{\sin u'} = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p}
\]

\[
E(x) = A(x) \cdot e^{i \phi(x)}
\]

\[
E(x) = A(x) \cdot e^{i k \Delta_{OPD}(x)}
\]

\[
E(x) = A(x) \cdot e^{2\pi i W(x)}
\]
Wave Aberration in Optical Systems

- Definition of optical path length in an optical system:
  Reference sphere around the ideal object point through the center of the pupil
- Chief ray serves as reference
  Difference of OPL : optical path difference OPD
- Practical calculation: discrete sampling of the pupil area,
  real wave surface represented as matrix

![Diagram](image)
Wave Aberration

- Exact relation between wave aberration and ray deviation
- General expression from geometry describes the lateral aberration
- Substitution of angle by scalar product
- Exact relation is quadratic in $R$
- Approximation for large $R$

$$
\Delta W = -\frac{x_p}{R} \cdot \Delta x' - \frac{y_p}{R} \cdot \Delta y' - \frac{x_p^2 + y_p^2}{2 \cdot R} \cdot \Delta z'
$$

with

$$
\bar{a} = \begin{pmatrix}
\Delta x' \\
\Delta y' \\
\Delta z'
\end{pmatrix}
$$

\[
W = W_0 + n' \cdot \left[ \frac{s_r - \bar{s} \cdot (\bar{s} \cdot s_r)}{1 + \bar{s} \cdot s_r} \right] \cdot \bar{a} + n' \cdot \frac{a^2 - (\bar{s} \cdot \bar{a})^2}{R \cdot (1 + \cos \theta)}
\]

\[
W = W_\infty + n' \cdot \frac{a^2 - (\bar{s} \cdot \bar{a})^2}{2R} \cdot \left[ 1 + \frac{a^2 - (\bar{s} \cdot \bar{a})^2}{4R^2} \right]
\]
Conversion Ray - Wave

- Rays and wavefronts are equivalent
  \[ E(x, y) = A(x, y) \cdot e^{2\pi i W(x, y)} \]
- Phase corresponds to ray direction
  \[ \vec{s} = \lambda \cdot \left( \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y} \right) \]
  \[ \frac{\partial W}{\partial x} = \frac{1}{\lambda} \cdot s_x, \quad \frac{\partial W}{\partial y} = \frac{1}{\lambda} \cdot s_y \]
- Amplitude A is described by ray weighting factor g,
  Transform of area element by Jacobian
  \[ g = \Delta A^2 \quad \Delta A' = \frac{\Delta A}{\left| \frac{\partial x'}{\partial x} \cdot \frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y} \cdot \frac{\partial y'}{\partial x} \right|} \]
Conversion Ray - Wave

- Realization in discrete ray and field sampling
  \[ x_p = x_Q + W(x_Q) \tan \varphi \]
  \[ \Delta A_j = \Delta x \cdot \Delta y \]
  \[ PQ = \frac{W(x_Q)}{\cos \varphi} \]

- Critical and limits of conversion in regions of a caustic,
  Failure of conversion for crossing of rays
Caustics

- Crossing rays
  - caustic surfaces
  - no unique ray direction
  - singular behavior of the eikonal with special solutions

- The singular solutions describes the envelope of the rays

- In the physical viewpoint, here interference takes place and the amplitude is no longer constant

- The caustic surface can be obtained from the eikonal equation by special techniques
Geometrical Optics Approximation

- Helmholtz wave equation
- Split into phase and amplitude
- The phase/optical path is given by
- Insertion, separation of real and imaginary part:
  1. equation
  2. equation
- Approximation of geometrical optics
  Gives the Eikonal equation for the description of ray propagation
  or with ray direction \( s \)
- Violation of the geometrical optical approximation:
  1. large values of \( \nabla A(\vec{r}) \) edges, diffraction takes place
  2. large values of \( \nabla L(\vec{r}) \) focal points, source points with large angles

\[
\Delta \tilde{E}(\vec{r}) + k^2 \cdot \tilde{E}(\vec{r}) = 0
\]
\[
E(\vec{r}) = A(\vec{r}) \cdot e^{-ik_oL(\vec{r})}
\]
\[
L = n \cdot \vec{s} \cdot \vec{r}
\]
\[
2\nabla A \cdot \nabla L + A \cdot \nabla^2 L = 0
\]
\[
k_o^2 \left( n^2(\vec{r}) - |\nabla L|^2 \right) \cdot A + \nabla^2 A = 0
\]
\[
a \gg \lambda \ , \ \lambda \cdot \nabla_x A \ll 1 \ , \ \frac{1}{k_o} \cdot \nabla_x A \ll 1
\]
\[
n^2(\vec{r}) - |\nabla L|^2 = 0
\]
\[
n(\vec{r}) \cdot \vec{s} = \nabla L
\]
Eikonal Equation

- Ansatz for Helmholtz wave equation with Eikonal $L$
  \[ E(\vec{r}) = E_o(\vec{r}) \cdot e^{-ik_o L(\vec{r})} \]

- Limiting case geometrical optic
  \[ \lambda_o \rightarrow 0 \]
  delivers the Eikonal equation:
  \[ (\nabla L)^2 = n^2(\vec{r}) \]

- $L$ describes the optical path length
  $L = \text{const.}$ are the phase fronts of the wave

- Application of the Eikonal equation:
  Numerical solution for the raytracing in inhomogeneous media (gradient)

- Complex $L$: evanescent damped waves
  \[ L = \int n(\vec{r}) \, d\vec{r} \]
Raytracing in Grin Media

- Ray: in general curved line in media of nonuniform refractive index

- Numerical solution of Eikonal equation by step-based Runge-Kutta algorithm, 4th order expansion, adaptive step size

- Analytical description of grin media by Taylor expansions of the function \( n(x,y,z) \)

\[
\begin{align*}
n &= n_{0,\lambda} + c_1 h + c_2 h^2 + c_3 h^4 + c_4 h^6 + c_5 h^8 + c_6 z + c_7 z^2 + c_8 z^3 + c_9 z^4 \\
&\quad + c_{10} x + c_{11} x^2 + c_{12} x^3 + c_{13} y + c_{14} y^2 + c_{15} y^3
\end{align*}
\]

- Large computational times necessary for high accuracy
Raytracing in Grin Media

- Numerical solution of the eikonal equation in case of nonhomogeneous media

1. step width \( \Delta t \)

2. scaled optical direction
\[ \vec{T} = n \cdot \vec{s} = \begin{pmatrix} n \cdot s_x \\ n \cdot s_y \\ n \cdot s_z \end{pmatrix} \]

3. new position and direction
\[ \vec{r}_{j+1} = \vec{r}_j + \Delta t \left( \vec{T}_j + \frac{\vec{A} + 2 \vec{B}}{6} \right) \]
\[ \vec{T}_{j+1} = \vec{T}_j + \frac{\vec{A} + 4 \vec{B} + \vec{C}}{6} \]

4. Runge-Kutta parameters 4th order
\[ \vec{A} = \Delta t \cdot \vec{D}(\vec{r}_j) \]
\[ \vec{B} = \Delta t \cdot \vec{D}\left( \vec{r}_j + \frac{\Delta t \cdot \vec{T}_j}{2} + \frac{\Delta t \cdot \vec{A}}{8} \right) \]
\[ \vec{C} = \Delta t \cdot \vec{D}\left( \vec{r}_j + \Delta t \cdot \vec{T}_j + \frac{\Delta t \cdot \vec{B}}{2} \right) \]
Gradient Lenses

- Refocusing in parabolic profile
- Helical ray path in 3 dimensions