Imaging and Aberration Theory

Lecture 14: Transfer function
2018-01-30
Herbert Gross
## Preliminary time schedule

<table>
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<th>Week</th>
<th>Date</th>
<th>Topic</th>
<th>Details</th>
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<tr>
<td>1</td>
<td>16.10</td>
<td>Paraxial imaging</td>
<td>Paraxial optics, fundamental laws of geometrical imaging, compound systems</td>
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<td>2</td>
<td>23.10</td>
<td>Pupils, Fourier optics, Hamiltonian coordinates</td>
<td>Pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates</td>
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<td>3</td>
<td>30.10</td>
<td>Eikonal</td>
<td>Fermat principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media</td>
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<td>4</td>
<td>06.11</td>
<td>Aberration expansions</td>
<td>Single surface, general Taylor expansion, representations, various orders, stop shift formulas</td>
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<td>5</td>
<td>13.11</td>
<td>Representation of aberrations</td>
<td>Different types of representations, fields of application, limitations and pitfalls, measurement of aberrations</td>
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<tr>
<td>6</td>
<td>20.11</td>
<td>Spherical aberration</td>
<td>Phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders</td>
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<tr>
<td>7</td>
<td>27.11</td>
<td>Distortion and coma</td>
<td>Phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options</td>
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<td>8</td>
<td>04.12</td>
<td>Astigmatism and curvature</td>
<td>Phenomenology, Coddington equations, Petzval law, correction options</td>
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<td>9</td>
<td>11.12</td>
<td>Chromatical aberrations</td>
<td>Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum</td>
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<td>10</td>
<td>18.12</td>
<td>Sine condition, aplanatism and isoplanatism</td>
<td>Sine condition, isoplanatism, relation to coma and shift invariance, pupil aberrations, Herschel condition, relation to Fourier optics</td>
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<tr>
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<td>Wave aberrations</td>
<td>Definition, various expansion forms, propagation of wave aberrations</td>
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<td>Zernike polynomials</td>
<td>Special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, measurement</td>
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<td>22.01</td>
<td>Point spread function</td>
<td>Ideal psf, psf with aberrations, Strehl ratio</td>
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<td>14</td>
<td>30.01</td>
<td>Transfer function</td>
<td>Transfer function, resolution and contrast</td>
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<td>15</td>
<td>05.02</td>
<td>Additional topics</td>
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Contents

- Fourier method
- Optical Transfer function
- Contrast and resolution
- Quantitative performance assessment
- Incoherent image formation
- Cascaded systems
- Coherent image formation
- 3D transfer theory and depth resolution
Definitions of Fourier Optics

- Phase space with spatial coordinate $x$ and
  1. angle $\theta$
  2. spatial frequency $\nu$ in mm$^{-1}$
  3. transverse wavenumber $k_x$

$$\theta_x = \lambda \cdot \nu = \frac{k_x}{k_0}$$

$$k = 2\pi \nu$$

- Fourier spectrum

$$A(v_x, v_y) = \hat{F}[E(x, y)]$$

corresponds to a plane wave expansion

$$A(k_x, k_y, z) = \iint E(x, y, z) e^{-i(xk_x + yk_y)} \, dx \, dy$$

- Diffraction at a grating with period $g$:
  deviation angle of first diffraction order varies linear with $\nu = 1/g$

$$\sin \theta = \frac{1}{g} = \lambda \cdot \nu$$
Arbitrary object expanded into a spatial frequency spectrum by Fourier transform

Every frequency component is considered separately

To resolve a spatial detail, at least two orders must be supported by the system

off-axis illumination

\[
g \cdot \sin \theta = m \cdot \lambda
\]

\[
g = \frac{\lambda}{\sin \theta} = \frac{\lambda}{NA}
\]

Ref: M. Kempe
Resolution of Fourier Components

- Ref: D. Aronstein / J. Bentley
Optical Transfer Function: Definition

- Normalized optical transfer function (OTF) in frequency space

\[
H_{OTF}(v_x, v_y) = N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{psf}(x_p, y_p) \cdot e^{-2\pi i (x_p v_x + y_p v_y)} \, dx_p \, dy_p
\]

- Fourier transform of the Psf-intensity

\[
H_{OTF}(v_x, v_y) = \hat{F}[I_{PSF}(x, y)]
\]

- OTF: Autocorrelation of shifted pupil function, Duffieux-integral (general: 2D)

\[
H_{OTF}(v_x) = \int P\left(x_p + \frac{\lambda f v_x}{2}\right) \cdot P^*\left(x_p - \frac{\lambda f v_x}{2}\right) \, dx_p
\]

- Transfer properties:
  PSF: response answer of a point object
  OTF: response answer of an extended cosine grating

- Absolute value of OTF: modulation transfer function (MTF)

- MTF is numerically identical to contrast of the image of a cosine grating at the corresponding spatial frequency
A structure of the object is resolved, if the first diffraction order is propagated through the optical imaging system.

The fidelity of the image increases with the number of propagated diffracted orders.
MTF and Contrast

- **Object**
  \[ I_{obj}(x) = c + a \cdot \cos(2\pi v_0 x) \]

- **Contrast**
  \[ V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{(c + a) - (c - a)}{(c + a) + (c - a)} = \frac{a}{c} \]

- **Object spectrum**
  \[ \tilde{I}_{obj}(v) = \hat{F}[I_{obj}(x)] = c \cdot \delta(v - 0) + \frac{a}{2} \cdot \delta(v - v_0) + \frac{a}{2} \cdot \delta(v + v_0) \]

- **Image spectrum**
  \[ \tilde{I}_{ima}(v) = \tilde{I}_{obj}(v) \cdot H_{MTF}(v) \]

- **Image**
  \[ I_{ima}(x') = \hat{F}^{-1}[\tilde{I}_{ima}(v)] = \hat{F}^{-1}[\tilde{I}_{obj}(v) \cdot H_{MTF}(v)] \]
  \[ = \hat{F}^{-1}\left[c \cdot H_{MTF}(v) \cdot \delta(v - 0) + \frac{a}{2} \cdot H_{MTF}(v) \cdot \delta(v - v_0) + \frac{a}{2} \cdot H_{MTF}(v) \cdot \delta(v + v_0)\right] \]
  \[ = c \cdot H_{MTF}(0) + \frac{a}{2} \cdot H_{MTF}(v_0) \cdot e^{2\pi iv_0 x} + \frac{a}{2} \cdot H_{MTF}(-v_0) \cdot e^{-2\pi iv_0 x} \]
  \[ = c + a \cdot H_{MTF}(v_0) \cdot \cos(2\pi v_0 x) \]
Interpretation of the Duffieux integral

- Interpretation of the Duffieux integral:
  - overlap area of 0th and 1st diffraction order, interference between the two orders

- The area of the overlap corresponds to the information transfer of the structural details

- Frequency limit of resolution:
  - areas completely separated

---

Interpretation of the Duffieux integral:
overlap area of 0th and 1st diffraction order, interference between the two orders

The area of the overlap corresponds to the information transfer of the structural details

Frequency limit of resolution:
areas completely separated
Duffieux Integral and Contrast

- Separation of pupils for 0. and ±1. Order
- MTF function
- Image contrast for sin-object

Ref: W. Singer
Resolution and Spatial Frequencies

- Grating object

- Imaging with NA = 0.8

- Imaging with NA = 1.3

Ref: L. Wenke
Optical Transfer Function of a Perfect System

- Aberration free circular pupil: Reference frequency
  \[ v_o = \frac{a}{\lambda f} = \frac{\sin u'}{\lambda} \]

- Cut-off frequency:
  \[ v_G = 2v_0 = \frac{2na}{\lambda f} = \frac{2n \sin u'}{\lambda} \]

- Analytical representation
  \[ H_{MTF}(v) = \frac{2}{\pi} \left[ \arccos \left( \frac{v}{2v_0} \right) - \left( \frac{v}{2v_0} \right) \sqrt{1 - \left( \frac{v}{2v_0} \right)^2} \right] \]

- Separation of the complex OTF function into:
  - absolute value: modulation transfer MTF
  - phase value: phase transfer function PTF
  \[ H_{OTF}(v_x, v_y) = H_{MTF}(v_x, v_y) \cdot e^{iH_{PTF}(v_x, v_y)} \]
Contrast / Visibility

- The MTF-value corresponds to the intensity contrast of an imaged sin grating
- Visibility

\[ V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} \]

- The maximum value of the intensity is not identical to the contrast value since the minimal value is finite too

- Concrete values:

<table>
<thead>
<tr>
<th>( \Delta I )</th>
<th>( I_{\text{max}} )</th>
<th>( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.010</td>
<td>0.990</td>
<td>0.980</td>
</tr>
<tr>
<td>0.020</td>
<td>0.980</td>
<td>0.961</td>
</tr>
<tr>
<td>0.050</td>
<td>0.950</td>
<td>0.905</td>
</tr>
<tr>
<td>0.100</td>
<td>0.900</td>
<td>0.818</td>
</tr>
<tr>
<td>0.111</td>
<td>0.889</td>
<td>0.800</td>
</tr>
<tr>
<td>0.150</td>
<td>0.850</td>
<td>0.739</td>
</tr>
<tr>
<td>0.200</td>
<td>0.800</td>
<td>0.667</td>
</tr>
<tr>
<td>0.300</td>
<td>0.700</td>
<td>0.538</td>
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</table>
Convolution of the object intensity distribution $I(x)$ changes:
1. Peaks are reduced
2. Minima are raised
3. Steep slopes are declined
4. Contrast is decreased
Due to the asymmetric geometry of the psf for finite field sizes, the MTF depends on the azimuthal orientation of the object structure.

Generally, two MTF curves are considered for sagittal/tangential oriented object structures.

![Diagram of MTF curves for sagittal and tangential orientations](image-url)
Real MTF of system with residual aberrations:
1. contrast decreases with defocus
2. higher spatial frequencies have stronger decrease
Polychromatic MTF

- Polychromatical MTF: Cut off frequency depends on $\lambda$
- Spectral incoherent weighted superposition of monochromatic MTF’s

$$g_{OTF}^{(poly)}(v) = \int_0^\infty S(\lambda) \cdot g_{OTF}(v, \lambda) \, d\lambda$$

![Graph showing polychromatic MTFs for different wavelengths](image)
Resolution/contrast criterion:
Ratio of contrasts with/without aberrations for one selected spatial frequency

\[ \Delta g_{MTF}(v) = \frac{g_{MTF}^{(real)}(v)}{g_{MTF}^{(ideal)}(v)} \]

Real systems:
Choice of several application relevant frequencies
e.g. photographic lens:
10 Lp/mm, 20 Lp/mm, 40 Lp/mm
Optical Transfer Function of a Perfect System

- Loss of contrast for higher spatial frequencies
Contrast vs contrast as a function of spatial frequency

- Typical: contrast reduced for increasing frequency

- Compromise between resolution and visibility is not trivial and depends on application
Contrast and Resolution

- High frequent structures: contrast reduced
- Low frequent structures: resolution reduced
Contrast / Resolution of Real Images

- Degradation due to
  1. loss of contrast
  2. loss of resolution

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Resolution, sharpness

Contrast, saturation
- Balance between contrast and resolution: not trivial
- Optimum depends on application
- Receiver: minimum contrast curve serves as real reference
  Most detector needs higher contrast to resolve high frequencies
- CSF: contrast sensitivity function

![Contrast vs Resolution Graph]

- \( g_{\text{MTF}} \)
  - \( 1 \) : high contrast
  - \( 2 \) : high resolution
  - Threshold contrast \( a \): 2 is better
  - Threshold contrast \( b \): 1 is better
- Photographic lenses with different performance
Resolution: Loss of Information

- Blurred imaging:
  - limiting case
  - information extractable

- Blurred imaging:
  - information is lost
  - what’s the time?
Geometrical Approximated Transfer Function

- Approximation of the transfer function for large aberrations:
  Expansion of the Duffieux-Integral

\[ g_{GTF}(v_x, v_y) = \iint \left[ 1 - \frac{1}{2} \left( v_x \cdot \Delta x + v_y \cdot \Delta y \right)^2 + i \left( v_x \cdot \Delta x + v_y \cdot \Delta y \right) \right] dx_p dy_p \]

- Approximation fails for large frequencies

- Example photographic lens
MTF for Aberrations
Phase Transfer Function

- OTF is complex function: decomposition into absolute value (MTF) and phase (PTF)
- Phase transfer function:
  - corresponds to a lateral offset of the image point
  - PTF describes distortion
  - can be used to assess coma aberration

\[ g_{OTF}(v_x, v_y) = g_{MTF}(v_x, v_y) \cdot e^{i \cdot g_{PTF}(v_x, v_y)} \]

\[ I_{PSF}(x) = I_{PSF}\left(x + \frac{g'_{PTF}(v)}{2\pi}\right) \]

spherical aberration: \( c_{40} = -0.25 \lambda \)
Cubic Phase Plate (CCP)

- Phase Mask with cubic polynomial shape

\[ P(x) = \begin{cases} 
  e^{i\alpha x^3} & \text{für } |x| \leq 1 \\
  0 & \text{sonst} 
\end{cases} \]

- Effect of mask:
  - depth of focus enlarged
  - Psf broadened, but nearly constant
  - Deconvolution possible

- Problems:
  - variable psf over field size
  - noise increased
  - finite chief ray angle
  - broadband spectrum in VIS
  - Image artefacts
Cubic Phase Mask: PSF and OTF

Conventional imaging

System with cubic phase mask

MTF eines beugungsbegrenzten optischen Systems

MTF von DeepView

Im Fokus
1 Schärftiefe defokussiert
2 Schärftiefe defokussiert

focus
defocussed

focus
defocussed
EDF Example Pictures

defocus negative

defocus positive

focussed
MTF / OTF for CPP

- Cubic phase plate
  - small changes of MTF with defocus
  - oscillation of real- and imaginary part:
    strong change of PTF

- The phase of the Fourier components is not properly reconstructed:
  artefacts in image deconvolution
MTF / OTF for CCP

- Real behavior of MTF:
  1. invariant MTF for lower spatial frequencies but re-magnification also increases noise
  2. decrease for higher spatial frequencies: reduction in resolution
Resolution Estimation with Test Charts

Measurement of resolution with test charts:
- bar pattern of different sizes
- two different orientations
- calibrated size/spatial frequency
Determination of resolution and contrast with Siemens star test chart:

- Central segments b/w
- Growing spatial frequency towards the center
- Gray ring zones: contrast zero
- Calibrating spatial feature size by radial diameter
- Nested gray rings with finite contrast in between: contrast reversal, pseudo resolution
Resolution Test Chart: Siemens Star

- **a.** original
- **b.** good system
- **c.** defocus
- **d.** spherical
- **e.** astigmatism
- **f.** coma
Fourier Optics – Point Spread Function

- Point spread function amplitude in an optical system with magnification $m$
  Pupil function $P$,
  Pupil coordinates $x_p, y_p$

\[
g_{psf}(x, y, x', y') = N \cdot \iiint P(x_p, y_p) \cdot e^{\frac{ik}{z} [x_p'(x' - mx) + y_p'(y' - my)]} \, dx_p \, dy_p
\]

- PSF is Fourier transform of the pupil function (scaled coordinates)

\[
g_{psf}(x, y) = N \cdot \hat{F}[P(x_p, y_p)]
\]
Fourier Theory of Incoherent Image Formation

- Transfer of an extended object distribution \( I_{\text{obj}}(x, y) \)

- In the case of shift invariance (isoplanatism):
  - incoherent convolution

- Intensities are additive

- In frequency space:
  - product of spectra
  - linear transfer theory
  - spectrum of the psf works as low pass filter onto the object spectrum

- Optical transfer function

\[
H_{\text{otf}}(v_x, v_y) = FT[I_{\text{PSF}}(x, y)]
\]

\[
I_{\text{image}}(v_x, v_y) = H_{\text{otf}}(v_x, v_y) \cdot I_{\text{obj}}(v_x, v_y)
\]

\[
I_{\text{ima}}(x', y') = \int \int I_{\text{psf}}(x, x', y, y') \cdot I_{\text{obj}}(x, y) \, dx \, dy
\]

\[
I_{\text{ima}}(x', y') = \int \int I_{\text{psf}}(x - x', y - y') \cdot I_{\text{obj}}(x, y) \, dx \, dy
\]

\[
I_{\text{ima}}(x', y') = I_{\text{psf}}(x, y) * I_{\text{obj}}(x, y)
\]
Incoherent Image Formation

- Example:
incoherent imaging of bar pattern near the resolution limit
Incoherent Image Formation

- Example:
  incoherent imaging of pattern near the resolution limit with aberrations

object

ideal

astigmatism

coma

spherical aberration

PSF
Cascaded Optical Systems

- Cascaded systems with individual transfer functions:
  1. Diffusing screen in intermediate image plane:
     Incoherent convolution of individual Psfs
     \[
     I_{\text{incoher}}(x) = I_{\text{psf}_2}(x) \otimes \left[ I_{\text{psf}_1}(x) \otimes I_{\text{object}}(x) \right]
     \]
     \[
     I_{\text{approx}}(x) = \left| A_{\text{psf}_1}(x) \right|^2 \otimes \left| A_{\text{psf}_2}(x) \right|^2
     \]
  2. Usual case:
     Second system works coherent
     Wave aberrations can be balanced
     Convolution of amplitude Psfs
     \[
     I_{\text{exact}}(x) = \left| A_{\text{psf}_1}(x) \otimes A_{\text{psf}_2}(x) \right|^2
     \]
Cascaded optical systems with individual transfer functions:

- Behaviour of the transfer functions in the limiting cases

**a) Single systems**

- \( \text{MTF}_1 \)
  - \( c_{\text{def}_1} = -c_{\text{def}_2} \)
  - Real
  - Perfect

- \( \text{MTF}_2 \)
  - \( c_{\text{def}_2} = -c_{\text{def}_1} \)
  - Real

**b) Complete cascaded system**

- \( \text{MTF}_{\text{coh}} \)
  - Correction
  - \( c_{\text{def}_1} = -c_{\text{def}_2} \)
  - Real
  - Coherent

- \( \text{MTF}_{\text{incoh}} \)
  - Superposed broadening
  - \( c_{\text{def}_1} \land c_{\text{def}_2} \)
  - Real
  - Incoherent
Cascaded Optical Systems

- Incoherent case:
  1. Psf convolution

\[
I_{\text{incoher}}(x) = I_{\text{psf}_2}(x) \otimes \left[ I_{\text{psf}_1}(x) \otimes I_{\text{object}}(x) \right]
\]

\[
I_{\text{incoher}}(x) = I_{\text{object}}(x) \otimes \left[ I_{\text{psf}_1}(x) \otimes I_{\text{psf}_2}(x) \right]
\]

2. MTF product

\[
I_{\text{incoher}}(v) = I_{\text{psf}_2}(v) \cdot \left[ I_{\text{psf}_1}(v) \cdot I_{\text{object}}(v) \right]
\]

\[
I_{\text{incoher}}(v) = I_{\text{object}}(v) \cdot \left[ I_{\text{psf}_1}(v) \cdot I_{\text{psf}_2}(v) \right]
\]
Cascaded Optical Systems

- Coherent case:
  1. Amplitude Psf convolution
     \[ I_{\text{exact}}(x) = \left| A_{\text{psf}1}(x) \otimes A_{\text{psf}2}(x) \right|^2 \]
     \[ = \left[ A_{\text{psf}1}(x) \otimes A_{\text{psf}2}(x) \right] \cdot \left[ A_{\text{psf}1}^*(x) \otimes A_{\text{psf}2}^*(x) \right] \]

  2. MTF product
     \[ I_{\text{exact}}(v) = \left[ T_1(v) \cdot e^{2\pi i W_1} \cdot T_2(v) \cdot e^{2\pi i W_2} \right] \otimes \left[ T_1(v) \cdot e^{-2\pi i W_1} \cdot T_2(v) \cdot e^{-2\pi i W_2} \right] \]
     \[ = \left[ T(v) \cdot e^{2\pi i W} \right] \otimes \left[ T(v) \cdot e^{-2\pi i W} \right] \]

---

object plane

system 1
~\( A_{\text{psf},1} \)

intermediate image

system 2
~\( A_{\text{psf},2} \)

image plane

\( C_{\text{def},1} \)

\( C_{\text{def},2} \)
Fourier Theory of Coherent Image Formation

- Transfer of an extended object distribution \( E_{obj}(x,y) \)

- In the case of shift invariance (isoplanasie):
  coherent convolution of fields

- Complex fields additive

- In frequency space:
  - product of spectra
  - linear transfer theory with fields
  - spectrum of the psf works as low pass filter onto the object spectrum
  - Coherent optical transfer function

\[
H_{ctf}(v_x, v_y) = FT\left[ E_{PSF}(x, y) \right]
\]

\[
E_{ima}(v_x, v_y) = H_{ctf}(v_x, v_y) \cdot E_{obj}(v_x, v_y)
\]

\[
I_{ima}(v_x, v_y) = \left| H_{ctf}(v_x, v_y) \cdot E_{obj}(v_x, v_y) \right|^2
\]

\[
E_{ima}(x', y') = \iint A_{psf}(x, y, x', y') \cdot E(x, y) \, dx \, dy
\]

\[
E_{ima}(x', y') = \iint A_{psf}(x - x', y - y') \cdot E_{obj}(x, y) \, dx \, dy
\]

\[
E_{ima}(x', y') = A_{psf}(x, y) * E_{obj}(x, y)
\]
Comparison of OTF and CTF

- Ideal coherent transfer function:
  - corresponds to scaled pupil function
  - cut off at pupil edge
  - full contrast until edge

- Ideal incoherent transfer function:
  - convolution of shifted scaled pupils
  - smooth decrease to cut-off frequency

- Incoherent case:
  - higher spatial frequencies resolved
  - lower contrast at low frequencies

\[
H_{ctf}(v) = P\left(\frac{x_p}{\lambda \cdot f}\right)
\]

\[
H_{otf}(v) = \frac{2}{\pi} \cdot \left[ \arccos\left(\frac{v}{2}\right) - \frac{v}{2} \cdot \sqrt{1 - \left(\frac{v}{2}\right)^2} \right]
\]
Ewald Sphere

- Assuming an object as grating with period $L$

$$\overrightarrow{k}_{\text{obj}} = \frac{2\pi}{L}$$

- Scattering of a wave at the object with
  - conservation of energy
  - conservation of momentum

- The outgoing $k$-vector must be on a sphere: Ewald's sphere for possible scattered wave vectors

$$|\overrightarrow{k}_{\text{in}}| = |\overrightarrow{k}_{\text{out}}|$$

$$\overrightarrow{k}_{\text{in}} + \overrightarrow{k}_{\text{obj}} = \overrightarrow{k}_{\text{out}}$$
McCutchen Formula and Axial Resolution

- Imaging of a plane wave at a volume object
  \( \delta x \): minimum value resolution
  \( \Delta v \): maximum interval
  Uncertainty relation: \( \Delta v \delta x = 1 \)

- Radius of the Ewald sphere
  generalized 3D pupil: red area

- Transverse resolution due to Abbe
  \[
  \delta x = \frac{2}{\Delta v_x} = \frac{2}{R \cdot \sin \theta} = \frac{2}{n / \lambda \cdot NA / n} = \frac{2\lambda}{NA}
  \]

- Axial resolution:
  - height of the cap of the cone
  - McCutchen formula
  \[
  \delta z = \frac{1}{\Delta v_z} = \frac{1}{R - R \cos \theta} = \frac{1}{n / \lambda \cdot \left(1 - \sqrt{1 - \sin^2 \theta}\right)}
  \]
  \[
  = \frac{\lambda}{n - \sqrt{n^2 - NA^2}} \approx \frac{2n\lambda}{NA^2}
  \]
3D Transfer Function

- Imaging as 3D scattering phenomenon
- Only special spatial frequencies are allowed due to energy conservation and momentum preservation
- Green circle: supported spatial frequencies of the transmitted wave vector

\[ \mathbf{v}_{\text{obj}} = \mathbf{v}_s - \mathbf{v}_i \]

\[ 2n/\lambda \]

\[ \mathbf{v}_z \]

\[ \mathbf{v}_x \]

\[ \mathbf{v}_s \]

\[ \mathbf{v}_{\text{obj}} \]

\[ \mathbf{v}_i \]

\[ \mathbf{v}_{\text{o-max}} \]

\[ \mathbf{v}_i \]

\[ \mathbf{v}_s \]

\[ \mathbf{v}_{\text{obj}} \]

\[ \mathbf{v}_{\text{o-max}} \]

backward

forward

Ewald sphere

Imaging as 3D scattering phenomenon

Only special spatial frequencies are allowed due to energy conservation and momentum preservation

Green circle: supported spatial frequencies of the transmitted wave vector
3D Transfer Function - Missing Cone

- Realistic case: finite numerical aperture
- Blue cone: possible incoming wave direction due to illumination cone
- 3D coherent transfer function: limited green area, that fulfills all conditions

- Missing cone: certain range of spatial axial spatial frequencies can not be seen in the image, Example: interfaces of thin coatings are not seen
3D Point Spread Function

- 3D Fourier transform of pupil with defocussing
  - $M : \text{magnification}$

\[
A_{psf}^{3D}(x, y) = \int \int P(x_p, y_p) \cdot e^{-\frac{ik}{2d^2}(z - M^2 \cdot z')} \left[ x_p^2 + y_p^2 \right] \cdot e^{\frac{ik}{d}(xx_p + yy_p)} \, dx_p \, dy_p
\]
General 3D Transfer Theory

- General coherent 3D transfer function
  Fourier transfer of psf field distribution
  \[
  H_{cft}(\vec{v}) = \int A_{psf}(\vec{r}') \cdot e^{2\pi i \vec{r} \cdot \vec{v}} \, d\vec{r}'
  \]

- Imaging of an transparent object with transmission function $T$
  \[
  E_{ima}(x', y', z') = e^{-ikz'} \cdot \iiint T_{obj}(v_x, v_y, v_z) \cdot H_{ctf}(v_x, v_y, v_z + \frac{1}{\lambda}) \cdot e^{-2\pi i \left( xv_x + yv_y - (v_z + \frac{1}{\lambda})M^2 z \right)} \, dv_x \, dv_y \, dv_z
  \]
  Offset $1 / \lambda$ : optical path length in thick object

- Special case of a single lens with pupil function $P$
  \[
  H_{ctf}^{rotsym}(v_r, v_z) = P(v_r) \cdot \delta \left( v_z - \frac{v_r^2}{2} \right)
  \]
  with
  \[
  v_{z0} = \frac{1}{\lambda \cdot \sin \theta_0}
  \]