### Preliminary time schedule

<table>
<thead>
<tr>
<th>Date</th>
<th>Topic</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.10</td>
<td>Paraxial imaging</td>
<td>Paraxial optics, fundamental laws of geometrical imaging, compound systems</td>
</tr>
<tr>
<td>07.11</td>
<td>Pupils, Fourier optics, Hamiltonian coordinates</td>
<td>Pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates</td>
</tr>
<tr>
<td>14.11</td>
<td>Eikonal</td>
<td>Fermat Principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media</td>
</tr>
<tr>
<td>21.11</td>
<td>Aberration expansion</td>
<td>Single surface, general Taylor expansion, representations, various orders, stop shift formulas</td>
</tr>
<tr>
<td>28.11</td>
<td>Representations of aberrations</td>
<td>Different types of representations, fields of application, limitations and pitfalls, measurement of aberrations</td>
</tr>
<tr>
<td>05.12</td>
<td>Spherical aberration</td>
<td>Phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders</td>
</tr>
<tr>
<td>12.12</td>
<td>Distortion and coma</td>
<td>Phenomenology, relation to sine condition, aplanatic sytems, effect of stop position, various topics, correction options</td>
</tr>
<tr>
<td>19.12</td>
<td>Astigmatism and curvature</td>
<td>Phenomenology, Coddington equations, Petzval law, correction options</td>
</tr>
<tr>
<td>09.01</td>
<td>Chromatical aberrations</td>
<td>Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum</td>
</tr>
<tr>
<td>16.01</td>
<td>Further reading on aberrations</td>
<td>Sensitivity in 3rd order, structure of a system, analysis of optical systems, lens contributions, Sine condition, isoplanatism, sine condition, Herschel condition, relation to coma and shift invariance, pupil aberrations, relation to Fourier optics</td>
</tr>
<tr>
<td>23.01</td>
<td>Wave aberrations</td>
<td>Definition, various expansion forms, propagation of wave aberrations, relation to PSF and OTF</td>
</tr>
<tr>
<td>30.01</td>
<td>Zernike polynomials</td>
<td>Special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, recalculation for offset, ellipticity, measurement</td>
</tr>
<tr>
<td>06.02</td>
<td>Miscellaneous</td>
<td>Intrinsic and induced aberrations, Aldi theorem, vectorial aberrations, partial symmetric systems</td>
</tr>
</tbody>
</table>
1. Definition of wave aberrations
2. Performance criteria
3. Primary aberrations
4. Order expansions
5. Non-circular pupil shapes
6. Statistical aberrations
7. Measurement of wave aberrations
Law of Malus-Dupin

- Law of Malus-Dupin:
  - equivalence of rays and wavefronts
  - both are orthonormal
  - identical information

- Condition:
  No caustic of rays

- Mathematical:
  Rotation of Eikonal vanish
  \[ \text{rot}(n \cdot \vec{s}) = 0 \]

- Optical system:
  Rays and spherical waves orthonormal

\[ \begin{align*}
L &= \text{const} \\
L &= \text{const}
\end{align*} \]
Relationships

- Concrete calculation of wave aberration: addition of discrete optical path lengths (OPL)

- Reference on chief ray and reference sphere (optical path difference)

- Relation to transverse aberrations

- Conversion between longitudinal transverse and wave aberrations

- Scaling of the phase / wave aberration:
  1. Phase angle in radiant
  2. Light path (OPL) in mm
  3. Light path scaled in $\lambda$

\[
l_{OPL}^{AP} = \int_{OE} n \cdot d\vec{r}
\]

\[
\Delta_{OPD}(x, y) = l_{OPL}(x, y) - l_{OPL}(0, 0)
\]

\[
\frac{\partial W}{\partial y_p} = -\frac{\Delta y'}{R - W} \approx \frac{\Delta y'}{R}
\]

\[
\Delta s' = \frac{R}{y_p} \cdot \Delta y' = \frac{\Delta y'}{\sin u'} = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p}
\]

\[
E(x) = A(x) \cdot e^{i \cdot \varphi(x)}
\]

\[
E(x) = A(x) \cdot e^{i \cdot k \Delta_{OPD}(x)}
\]

\[
E(x) = A(x) \cdot e^{2 \pi i \cdot W(x)}
\]
Relationship to Transverse Aberration

- Relation between wave and transverse aberration
- Approximation for small aberrations and small aperture angles $u$
- Ideal wavefront, reference sphere: $W_{\text{ideal}}$
- Real wavefront: $W_{\text{real}}$
- Finite difference
  \[ \Delta W = W_{\text{real}} - W_{\text{ideal}} \]
- Angle difference
- Transverse aberration
- Limiting representation

\[
\varphi \approx \tan \varphi = \frac{\partial W}{\partial y_p}
\]

\[
\Delta y' = -R \cdot \varphi
\]

\[
\frac{\partial W}{\partial y_p} = -\frac{\Delta y'}{R-W} \approx -\frac{\Delta y'}{R}
\]
Wave Aberration

- Exact relation between wave aberration and ray deviation
- General expression from geometry
  - a describes the lateral aberration
- Substitution of angle by scalar product
- Exact relation is quadratic in R
- Approximation for large R

\[
W = W_0 + n' \left( \frac{\hat{s}_r - \hat{s} \cdot (\hat{s} \cdot \hat{s}_r)}{1 + \hat{s} \cdot \hat{s}_r} \right) \cdot \hat{a} + n' \frac{a^2 - (\hat{s} \cdot \hat{a})^2}{R \cdot (1 + \cos \theta)}
\]

\[
W = W_\infty + n' \frac{a^2 - (\hat{s} \cdot \hat{a})^2}{2R} \left[ 1 + \frac{a^2 - (\hat{s} \cdot \hat{a})^2}{4R^2} \right]
\]

\[
\Delta W = -\frac{x_p}{R} \Delta x' - \frac{y_p}{R} \Delta y' - \frac{x_p^2 + y_p^2}{2 \cdot R} \Delta z'
\]

with

\[
\hat{a} = \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}
\]
Wave Aberration in Optical Systems

- Definition of optical path length in an optical system:
  Reference sphere around the ideal object point through the center of the pupil
- Chief ray serves as reference
  Difference of OPL : optical path difference OPD
- Practical calculation: discrete sampling of the pupil area, real wave surface represented as matrix
Pupil Sampling

- All rays start in one point in the object plane
- The entrance pupil is sampled equidistant
- In the exit pupil, the transferred grid may be distorted
- In the image plane a spreaded spot diagram is generated
Wave Aberration

- Definition of the peak valley value $W_{PV}$
- Reference sphere corresponds to perfect imaging
- Rms-value is more relevant for performance evaluation
Wave Aberration Criteria

- Mean quadratic wave deviation (\( W_{\text{Rms}} \), root mean square)

\[
W_{\text{rms}} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \sqrt{\frac{1}{A_{\text{ExP}}} \iint \left[ W(x_p, y_p) - W_{\text{mean}}(x_p, y_p) \right]^2 dx_p dy_p}
\]

with pupil area

\[
A_{\text{ExP}} = \iint dx dy
\]

- Peak valley value \( W_{pv} \): largest difference

\[
W_{pv} = \max \left[ W_{\text{max}}(x_p, y_p) - W_{\text{min}}(x_p, y_p) \right]
\]

- General case with apodization: weighting of local phase errors with intensity, relevance for psf formation

\[
W_{\text{rms}} = \sqrt{\frac{1}{A_{\text{ExP}}^{(w)}}} \iint I_{\text{ExP}}(x_p, y_p) \cdot \left[ W(x_p, y_p) - W_{\text{mean}}^{(w)}(x_p, y_p) \right]^2 dx_p dy_p
\]
Wave Aberrations – Sign and Reference

- Wave aberration: relative to reference sphere
- Choice of offset value: vanishing mean
- Sign of $W$:
  - $W > 0$: stronger convergence, intersection: $s < 0$
  - $W < 0$: stronger divergence, intersection: $s < 0$

$$\langle W(x, y) \rangle = \frac{1}{F_{Exp}} \iint W(x, y) \, dx \, dy = 0$$
Tilt of Wavefront

- Change of reference sphere:
  tilt by angle $\theta$
  linear in $y_p$
  $\Delta W_{\text{tilt}} = n \cdot y_p \cdot \theta$

- Wave aberration due to transverse aberration $\Delta y'$
  $\Delta W_{\text{tilt}} = -\frac{y_p}{R_{\text{Ref}}} \cdot \Delta y'$

- Is the usual description of distortion

![Diagram of wavefront with tilt angle $\theta$, pupil plane, wave front, reference sphere, and image plane.](image)
Defocussing of Wavefront

Paraxial defocussing by $\Delta z$:
Change of wavefront

$$\Delta W_{Def} = -\frac{n \cdot r_p^2}{2R_{ref}^2} \cdot \Delta z' = -\frac{1}{2} n \cdot \Delta z' \cdot \sin^2 \theta$$
Primary Aberrations

- Representation of primary aberrations
  - Seidel terms
- Surface in pupil plane
- Special case of chromatical aberrations

### Primary monochromatic wave aberrations

<table>
<thead>
<tr>
<th></th>
<th>Spherical aberration</th>
<th>Coma</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W = c_1 \cdot (x_p^2 + y_p^2) = c_1 \cdot r^4$</td>
<td>$W = c_2 \cdot y_p \cdot (x_p^2 + y_p^2) = c_2 \cdot yr^3 \cos \varphi$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Astigmatism</th>
<th>Field curvature (sagittal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W = c_3 \cdot y^2 y_p = c_3 \cdot y^2 r^2 \cos^2 \varphi$</td>
<td>$W = c_4 \cdot y^2 \cdot (x_p^2 + y_p^2) = c_4 \cdot y^2 r^2$</td>
<td></td>
</tr>
</tbody>
</table>

### Primary chromatic wave aberrations

<table>
<thead>
<tr>
<th></th>
<th>Axial color</th>
<th>Lateral color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta W = \vec{b_1} \cdot (x_p^2 + y_p^2) = \vec{b_1} \cdot r^2$</td>
<td>$\delta W = \vec{b_2} \cdot y_p = \vec{b_2} \cdot yr \cos \varphi$</td>
<td></td>
</tr>
</tbody>
</table>

Ref: H. Zügge
Primary Aberrations

- Relation: wave / geometrical aberration

<table>
<thead>
<tr>
<th>Type</th>
<th>Wave aberration</th>
<th>Geometrical spot</th>
</tr>
</thead>
</table>
| Spherical aberration          | $W = c_1 \cdot r^4$ | $\Delta x' \propto c_1 \cdot r^3 \sin \phi$
| Symmetry to Periodicity       |                 | $\Delta y' \propto c_1 \cdot r^3 \cos \phi$
| Symmetry axis constant point  | $x_p$          | point 1 period                    |
| Coma                          | $W = c_2 \cdot yr \cos \varphi$ | $\Delta y' \propto c_2 \cdot yr^2 \cdot (2 + \cos 2\varphi)$
| Symmetry to Periodicity       | one plane       | one straight line 2 periods       |
| Symmetry one plane 1 period   | $y_p$          |                                   |
| Astigmatism                   | $W = c_3 \cdot y^2r^2 \cos^2 \varphi$ | $\Delta x' = 0$
| Symmetry to Periodicity       | two planes      | two straight lines 1 period       |
| Symmetry two planes 2 period  | $y_p$          |                                   |
| Field curvature (sagittal)    | $W = c_4 \cdot y^2r^2$ | $\Delta x' \propto c_4 \cdot y^2r \cos \varphi$
| Symmetry to Periodicity       | axis constant   |                                   |
| Symmetry axis constant point  | $x_p$          | point 1 period                    |
| Distortion                    | $W = c_5 \cdot y^3r \cos \varphi$ | $\Delta x' = 0$
| Symmetry to Periodicity       | one plane       | one straight line constant        |
| Symmetry one plane 1 period   | $y_p$          |                                   |
Primary Aberrations

- **Relation:**
  wave / geometrical aberration

<table>
<thead>
<tr>
<th>Type</th>
<th>Wave aberration</th>
<th>Geometrical spot</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Axial color</strong></td>
<td>( \delta W = \tilde{b}_1 \cdot r^2 )</td>
<td>( \Delta x' \propto \tilde{b}_1 \cdot r \sin \phi ) ( \Delta y' \propto \tilde{b}_1 \cdot r \cos \phi )</td>
</tr>
<tr>
<td>Symmetry to</td>
<td>axis constant</td>
<td>point 1 period</td>
</tr>
<tr>
<td>Periodicity</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Lateral color</strong></td>
<td>( \delta W = \tilde{b}_2 \cdot yr \cos \phi )</td>
<td>( \Delta x' = 0 ) ( \Delta y' \propto \tilde{b}_2 \cdot y )</td>
</tr>
<tr>
<td>Symmetry to</td>
<td>one plane 1 period</td>
<td>one straight line constant</td>
</tr>
<tr>
<td>Periodicity</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ref: H. Zügge
Special Cases of Wave Aberrations

- Wave aberrations are usually given as reduced aberrations:
  - wave front for only 1 field point
  - field dependence represented by discrete cases

- Special case of aberrations:
  1. axial color and field curvature: represented as defocussing term, Zernike $c_4$

  2. distortion and lateral color: represented as tilt term, Zernike $c_2$, $c_3$
3. afocal system
   - exit pupil in infinity
   - plane wave as reference

4. telecentric system
   chief ray parallel to axis
## Expansion of the Wave Aberration

- Table as function of field and aperture
- Selection rules: checkerboard filling of the matrix

<table>
<thead>
<tr>
<th>Aperture r</th>
<th>Field y</th>
<th>Spherical</th>
<th>Coma</th>
<th>Astigmatism</th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>y^0</td>
<td>y^1</td>
<td>y^2</td>
<td>y^3</td>
</tr>
<tr>
<td></td>
<td>Distortion</td>
<td>r^1</td>
<td>Y r cosθ Tilt</td>
<td>y^3 r cosθ Distortion primary</td>
<td>y^5 r cosθ Distortion secondary</td>
</tr>
<tr>
<td></td>
<td>r^2</td>
<td>r^2</td>
<td>r^2 cos^2θ</td>
<td>y^2 r^2 cos^2θ</td>
<td>y^4 r^2 cos^2θ</td>
</tr>
<tr>
<td></td>
<td>r^3</td>
<td>r^3</td>
<td>y^3 cosθ Coma primary</td>
<td>y^3 r^3 cos^3θ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>r^4</td>
<td>r^4</td>
<td>y^4 cos^2θ</td>
<td>y^2 r^4 cos^2θ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>r^5</td>
<td>r^5</td>
<td>y^5 cosθ Coma secondary</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>r^6</td>
<td>r^6</td>
<td>y^6 cosθ Coma secondary</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Secondary aberrations:

Primary aberrations / Seidel

Image location
Polynomial Expansion of Wave Aberrations

- Taylor expansion of the wavefront:
  \[ W(y', r_p, \theta) = \sum_{k,l,m} W_{klm} y'^k r_p^l \cos^m \theta \]
  - \( y' \) Image height
  - \( r_p \) Pupil height
  - \( \theta \) Pupil azimuth angle

- Symmetry invariance:
  1. Image height
  2. Pupil height
  3. Scalar product between image and pupil vector

- Number of terms
  sum of indices in the exponent \( i_{\text{sum}} \)

<table>
<thead>
<tr>
<th>( i_{\text{sum}} )</th>
<th>( N_i ), number of terms</th>
<th>Type of aberration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>image location</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>primary aberrations, 4th order</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>secondary aberrations, 6th order</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>8th order</td>
</tr>
</tbody>
</table>
The exponents of the Taylor expansion on the aperture depends on the kind of representation of the aberrations.

The exponent grows by 1 in the sequence longitudinal-transversal-wave aberrations.

The Seidel term '3rd order' is valid only for transverse aberrations.

Dependence on aperture and field size for the primary aberrations:

<table>
<thead>
<tr>
<th>type of aberration</th>
<th>wave aberration</th>
<th>transverse aberration</th>
<th>longitudinal aberration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>u</td>
<td>w</td>
<td>u</td>
</tr>
<tr>
<td>spherical</td>
<td>4</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>coma</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>astigmatism</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Petzval curvature</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>distortion</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>axial chromatical</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>chromatical magnif.</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Taylor Expansion of the Primary Aberrations

- Expansion of the monochromatic aberrations
- First real aberration: primary aberrations, 4th order as wave deviation

\[ W(y', r_p, y_p) = A_S r_p^4 + A_C y' r_p^2 y_p + A_A y'^2 y_p^2 + A_P y'^2 r_p^2 + A_D y'^3 y_p \]

- Coefficients of the primary aberrations:
  \( A_S \): Spherical Aberration
  \( A_C \): Coma
  \( A_A \): Astigmatism
  \( A_P \): Petzval curvature
  \( A_D \): Distortion

- Alternatively: expansion in polar coordinates:
  Zernike basis expansion, usually only for one field point, orthogonalized
Criteria of Rayleigh and Marechal

- Rayleigh criterion:
  1. maximum of wave aberration: $W_{pv} < \lambda/4$
  2. beginning of destructive interference of partial waves
  3. limit for being diffraction limited (definition)
  4. as a PV-criterion rather conservative: maximum value only in 1 point of the pupil
  5. different limiting values for aberration shapes and definitions (Seidel, Zernike,...)

- Marechal criterion:
  1. Rayleigh crierion corresponds to $W_{\text{rms}} < \lambda/14$ in case of defocus
     \[
     W_{\text{Rayleigh}}^{\text{rms}} \leq \frac{\lambda}{\sqrt{192}} = \frac{\lambda}{13.856} \approx \frac{\lambda}{14}
     \]
  2. generalization of $W_{\text{rms}} < \lambda/14$ for all shapes of wave fronts
  3. corresponds to Strehl ratio $D_s > 0.80$ (in case of defocus)
  4. more useful as PV-criterion of Rayleigh
Rayleigh Criterion

- The Rayleigh criterion \( |W_{pv}| \leq \frac{\lambda}{4} \)
- The Rayleigh criterion gives individual maximum aberrations coefficients, depends on the form of the wave

- Examples:

<table>
<thead>
<tr>
<th>aberration type</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus Seidel</td>
<td>( a_{20} = 0.25 )</td>
</tr>
<tr>
<td>defocus Zernike</td>
<td>( c_{20} = 0.125 )</td>
</tr>
<tr>
<td>spherical aberration Seidel</td>
<td>( a_{40} = 0.25 )</td>
</tr>
<tr>
<td>spherical aberration Zernike</td>
<td>( c_{40} = 0.167 )</td>
</tr>
<tr>
<td>astigmatism Seidel</td>
<td>( a_{22} = 0.25 )</td>
</tr>
<tr>
<td>astigmatism Zernike</td>
<td>( c_{22} = 0.125 )</td>
</tr>
<tr>
<td>coma Seidel</td>
<td>( a_{31} = 0.125 )</td>
</tr>
<tr>
<td>coma Zernike</td>
<td>( c_{31} = 0.125 )</td>
</tr>
</tbody>
</table>

a) optimal constructive interference
b) reduced constructive interference due to phase aberrations
c) reduced effect of phase error by apodization and lower energetic weighting
d) start of destructive interference for 90° or \( \lambda/4 \) phase aberration begin of negative z-component
PV and $W_{\text{rms}}$-Values

- PV and $W_{\text{rms}}$ values for different definitions and shapes of the aberrated wavefront

- Due to mixing of lower orders in the definition of the Zernikes, the $W_{\text{rms}}$ usually is smaller in comparison to the corresponding Seidel definition

<table>
<thead>
<tr>
<th>aberration type</th>
<th>definition</th>
<th>mean $W_{\text{mean}}$</th>
<th>peak-valley $W_{\text{PV}}$</th>
<th>root mean square $W_{\text{rms}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus</td>
<td>$a_{20} \cdot r_p^2$</td>
<td>$\frac{a_{20}}{2}$</td>
<td>$a_{20}$</td>
<td>$\frac{a_{20}}{2 \sqrt{3}} = 0.289 \cdot a_{20}$</td>
</tr>
<tr>
<td>defocus</td>
<td>$c_{20} \cdot (2r_p^2 - 1)$</td>
<td>0</td>
<td>$2c_{20}$</td>
<td>$\frac{c_{20}}{\sqrt{3}} = 0.577 \cdot c_{20}$</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>$a_{40} \cdot r_p^4$</td>
<td>$\frac{a_{40}}{3}$</td>
<td>$a_{40}$</td>
<td>$\frac{2a_{40}}{3 \sqrt{5}} = 0.298 \cdot a_{40}$</td>
</tr>
<tr>
<td>spherical aberration with defocus</td>
<td>$b_{40} \cdot (r_p^4 - r_p^2)$</td>
<td>$-\frac{b_{40}}{6}$</td>
<td>$\frac{b_{40}}{4}$</td>
<td>$\frac{b_{40}}{6 \sqrt{5}} = 0.075 \cdot b_{40}$</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>$c_{40} \cdot (6r_p^4 - 6r_p^2 + 1)$</td>
<td>0</td>
<td>$\frac{3c_{40}}{5}$</td>
<td>$\frac{c_{40}}{\sqrt{5}} = 0.447 \cdot c_{40}$</td>
</tr>
<tr>
<td>astigmatism</td>
<td>$a_{22} r_p^2 \cos \theta$</td>
<td>$\frac{a_{22}}{4}$</td>
<td>$a_{22}$</td>
<td>$\frac{a_{22}}{4} = 0.25 \cdot a_{22}$</td>
</tr>
<tr>
<td>astigmatism with defocus</td>
<td>$b_{22} \left(r_p^2 \cos \theta - \frac{1}{2} r_p^2\right)$</td>
<td>0</td>
<td>$b_{22}$</td>
<td>$\frac{b_{22}}{2 \sqrt{6}} = 0.204 \cdot b_{22}$</td>
</tr>
<tr>
<td>astigmatism</td>
<td>$c_{22} \left(2r_p^2 \cos \theta - r_p^2\right)$</td>
<td>0</td>
<td>$2c_{22}$</td>
<td>$\frac{c_{22}}{\sqrt{6}} = 0.408 \cdot c_{22}$</td>
</tr>
<tr>
<td>coma</td>
<td>$a_{31} r_p^3 \cos \theta$</td>
<td>0</td>
<td>$2a_{31}$</td>
<td>$\frac{a_{31}}{2 \sqrt{2}} = 0.353 \cdot a_{31}$</td>
</tr>
<tr>
<td>coma with tilt</td>
<td>$b_{31} \left(r_p^3 - \frac{2}{3} r_p^2\right) \cos \theta$</td>
<td>0</td>
<td>$\frac{2b_{31}}{3}$</td>
<td>$\frac{b_{31}}{6 \sqrt{2}} = 0.118 \cdot b_{31}$</td>
</tr>
<tr>
<td>coma</td>
<td>$c_{31} \left(3r_p^3 - 2r_p^2\right) \cos \theta$</td>
<td>0</td>
<td>$2c_{31}$</td>
<td>$\frac{c_{31}}{2 \sqrt{2}} = 0.353 \cdot c_{31}$</td>
</tr>
</tbody>
</table>
Typical Variation of Wave Aberrations

- Microscopic objective lens

- Changes of rms value of wave aberration with
  1. wavelength
  2. field position

- Common practice:
  1. diffraction limited on axis for main part of the spectrum
  2. Requirements relaxed in the outer field region
  3. Requirement relaxed at the blue edge of the spectrum
Zernike Polynomials

- Expansion of wave aberration surface

\[ W(r, \varphi) = \sum_{n} \sum_{m=-n}^{n} c_{nm} Z_{n}^{m}(r, \varphi) \]

- Zernike polynomials orders by indices:
  - \( n \) : radial
  - \( m \) : azimuthal, sin/cos
- Orthonormal function on unit circle

\[ Z_{n}^{m}(r, \varphi) = R_{n}^{m}(r) \cdot \begin{cases} 
\sin(m\varphi) & \text{for } m < 0 \\
\cos(m\varphi) & \text{for } m > 0 \\
1 & \text{for } m = 0 
\end{cases} \]

- Direct relation to primary aberration types
- Direct measurement by interferometry
- Orthogonality perturbed:
  1. apodization
  2. discretization
  3. real non-circular boundary
- Orthogonalization of Zernike Polynomials for ring shaped pupil area

- Basis function depends on obsuration parameter $e$: no easy comparisons possible
Systems with rectangular pupil:
Use of Legendre polynomials $P_n(x)$

1. Factorized representation
   Problem: zero-crossing lines
2. Definition of 2D area-orthogonal
   Legendre functions

General shape of the pupil area:
Gram-Schmidt-orthogonalization
drawback:
1. Individual function for every pupil shape
2. no intuitive interpretation
3. no comparability between different systems possible

\[
\int_{-1}^{+1} P_n(x) \cdot P_m(x) \, dx = \begin{cases} 
0 & \text{if } n \neq m \\
\frac{2}{2n+1} & \text{if } n = m
\end{cases}
\]

\[
W(x, y) = \sum_{n} \sum_{m} A_{nm} P_n(x) \cdot P_m(y)
\]
Orthogonalization of Zernike polynomials on a unit square

\[ \int_{-1}^{1} \int_{-1}^{1} Q_k Q_j \, dx \, dy = \delta_{kj} \]

\[ Q_m = \sum_{j=0}^{j_{\text{max}}} \sum_{l=0}^{l_{\text{max}}} d_{m, jl} \cdot x^j y^l \]

\[ Q_m = \sum_{j=1}^{m} c_{mj} \cdot Z_j \]

Gram-Schmidt-Orthogonalization procedure

\[ Q_m = \sum_{k=1}^{m-1} a_k \cdot Q_k + a_m \cdot Z_m \]

\[ c_{mk} = \sum_{j=k}^{m-1} a_j \cdot c_{jk} \quad k = 1, 2, 3, \ldots m - 1 \]

\[ a_m = c_{mm} = \frac{1}{\sqrt{P_{mm} + \sum_{k=1}^{m-1} T_{km}^2 - 2 \sum_{k=1}^{m-1} T_{km} \sum_{j=1}^{k} c_{kj} \cdot P_{mj}}} \]
Legendre Polynomials

- 2D-Legendre polynomials for rectangular areas
- Application: Spectrometer slit aperture
- First few polynomials: quite similar to Zernikes
Statistical Wave Aberrations

- Complex field with statistical phase
  \[ E(\vec{r}) = E_o(\vec{r})e^{i\Phi(\vec{r})} \]

- Correlation of the phase: structural function
  \[ e^{-\frac{1}{2}D_\Phi(r_{12})} = \langle e^{i\Phi(\vec{r}_1) - i\Phi(\vec{r}_2)} \rangle \]

- Coherence function
  \[ \Gamma_{12}(z) = E_o(\vec{r}_1)E_o(\vec{r}_2)e^{-\frac{1}{2}D_\Phi(r_{12})} \]

- For gaussian statistics

- Auto covariance function

- PSD, power spectral density

\[ C(\Delta x, \Delta y) = \langle \Phi(x, y)\Phi(x + \Delta x, y + \Delta y) \rangle = \sigma^2 e^{-\frac{x^2+y^2}{a_c^2}} \]

\[ S(v_x, v_y) = \hat{F}[C(x, y)] = \pi\sigma^2 a_c^2 e^{-\pi^2 a_c^2(v_x^2+v_y^2)} \]
Statistical Wave Aberrations

- Description:
  1. in the spatial domain: topology of the rough surface
  2. in the spatial frequency domain

\[ h(x) \]
\[ C(x) \]
\[ A(\nu) \]
\[ \text{PSD}(\nu) \]
\[ \text{Fourier transform} \]
\[ < h_1 h_2 > \]
\[ \text{correlation} \]
\[ \text{amplitude spectrum} \]
\[ \text{power spectral density} \]
\[ | |^2 \]
\[ \text{square} \]
- Power spectral density of the perturbation
- Three typical frequency ranges, scaled by diameter $D$

![Graph showing the power spectral density of surface perturbations with three frequency ranges labeled: long range, low frequency, mid frequency, and micro roughness.](image)
Atmospheric Turbulence

- Atmospheric turbulence: statistical phase screen

- Scale below 1 cm: Tatarski regime, viscosity PSD
  \[ \Phi = b_{Ta} \cdot e^{-\left( \frac{k}{k_i} \right)^2} \]

- Scale from 1 cm to 5 m: Kolmogorov regime PSD
  \[ \Phi = b_{Ta} \cdot k^{\frac{-11}{3}} \]

- Greater length scales: Karman regime PSD
  \[ \Phi = b_{Ka} \cdot \left( k^2 + \frac{4\pi^2}{L_o^2} \right)^{\frac{-11}{6}} \]
Measurement of Wave Aberrations

- Wave aberrations are measurable directly
- Good connection between simulation/optical design and realization/metrology
- Direct phase measuring techniques:
  1. Interferometry
  2. Hartmann-Shack
  3. Hartmann sensor
  4. Special: Moire, Holography, phase-space analyzer
- Indirect measurement by inversion of the wave equation:
  1. Phase retrieval of PSF z-stack
  2. Retrieval of edge or line images
- Indirect measurement by analyzing the imaging conditions: from general image degradation
- Accuracy:
  1. $\lambda/1000$ possible, $\lambda/100$ standard for rms-value
  2. Rms vs. individual Zernike coefficients
Testing with Twyman-Green Interferometer

- Short common path, sensible setup
- Two different operation modes for reflection or transmission
- Always factor of 2 between detected wave and component under test

1. mode: lens tested in transmission auxiliary mirror for auto-collimation
2. mode: surface tested in reflection auxiliary lens to generate convergent beam
Interferograms of Primary Aberrations

Spherical aberration $1\lambda$

-1 -0.5 0 +0.5 +1

Defocussing in $\lambda$

Astigmatism $1\lambda$

Coma $1\lambda$
Problems in real world measurement:

- **Edge effects**
  - Definition of boundary

- **Perturbation by coherent stray light**

- **Local surface error are not well described by Zernike expansion**

- **Convolution with motion blur**

Ref: B. Dörband
Critical definition of the interferogram boundary and the Zernike normalization radius in reality
- Wave front determines local direction of propagation
- Propagation over distance $z$: change of transverse intensity distribution
- Intensity propagation contains phase information

**Principle of Psf Phase Retrieval**

wave front: $W(x,y)$
local Poynting vector
direction of propagation

intensity caustic
$I(x,y,z)$

propagation

$z$
Transport of intensity equation couples phase and intensity

\[ k \cdot \frac{\partial I(x, y, z)}{\partial z} = - \nabla[I(x, y, z) \cdot \nabla W(x, y)] \]

Solution with z-variation of the intensity delivers start phase at \( z = 0 \)

Determine phase from intensity distribution.
- Inverse propagation problem: ill posed
- Boundary condition: measured z-stack \( I(x, y, z) \)

Algorithm for numerical solution
- IFTA / Gerchberg Saxton (error reduction)
- Acceleration (conjugate gradients, Fienup, ...)
- Modal non least square methods
- Extended Zernike method

Applications:
- Calculation of diffractive components for given illumination distribution
- Wave front reconstruction
- Phase microscopy
- Principle of phase retrieval for metrology of optical systems
- Measurement of intensity caustic z-stack
- Reconstruction of the phase in the exit pupil
Gerchberg-Saxton-Algorithm

- Iterative reconstruction of the pupil phase with back-and-forth calculation between image and pupil: IFTA / Gerchberg-Saxton

- Substitution of known intensity

- Problems with convergence: Twin-image degeneration

- Modified algorithms:
  1. Fienup-acceleration
  2. Non-least-square
  3. Use of pupil intensity
Phase Space Interpretation

- Known measurement of intensity in defocussed planes:
  - Several rotated planes in phase space
  - Information in and near the spatial domain
- Calculation of distribution in the Fourier plane
- Wave equation is valid
- Principle: Tomography
Example Phase Retrieval

- Evaluation of real data psf-stack
Phase Retrieval with Apodization

- Retrieval without / with Apodization
- Correlation over z
Object Space Defocussing

a) defocussing in image space

b) defocussing in object space
Measurements

- Comparison phase retriev vs Hartmann test
- Case of coma

Ref: B. Möller