Imaging and Aberration Theory

Lecture 2: Pupil, Fourier optics and Hamiltonians
2013-10-31
Herbert Gross
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<td>24.10</td>
<td>Paraxial imaging</td>
<td>Paraxial optics, fundamental laws of geometrical imaging, compound systems</td>
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<td>Pupils, Fourier optics, Hamiltonian coordinates</td>
<td>Pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates</td>
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<td>Eikonal</td>
<td>Fermat Principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media</td>
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<td>Aberration expansion</td>
<td>Single surface, general Taylor expansion, representations, various orders, stop shift formulas</td>
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<td>Representations of aberrations</td>
<td>Different types of representations, fields of application, limitations and pitfalls, measurement of aberrations</td>
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<td>Spherical aberration</td>
<td>Phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders</td>
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<td>Distortion and coma</td>
<td>Phenomenology, relation to sine condition, aplanatic sytems, effect of stop position, various topics, correction options</td>
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<td>Astigmatism and curvature</td>
<td>Phenomenology, Coddington equations, Petzval law, correction options</td>
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<td>19.12</td>
<td>Chromatical aberrations</td>
<td>Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary sppectrum</td>
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<td>Further reading on aberrations</td>
<td>Sensitivity in 3rd order, structure of a system, analysis of optical systems, lens contributions, Sine condition, isoplanatism, sine condition, Herschel condition, relation to coma and shift invariance, pupil aberrations, relation to Fourier optics</td>
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<td>Wave aberrations</td>
<td>Definition, various expansion forms, propagation of wave aberrations, relation to PSF and OTF</td>
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<td>Zernike polynomials</td>
<td>Special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, recalculation for offset, ellipticity, measurement</td>
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<td>13</td>
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<td>Miscellaneous</td>
<td>Intrinsic and induced aberrations, Aldi theorem, telecentric case, afocal case, aberration balancing, Scheimpflug imaging, Fresnel lenses, statistical aberrations</td>
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<td>14</td>
<td>06.02</td>
<td>Vectorial aberrations</td>
<td>Introduction, special cases, actual research, anamorphic, partial symmetric</td>
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1. Definition of aperture and pupil
2. Special ray sets
3. Vignetting
4. Helmholtz-Lagrange invariant
5. Phase space
6. Resolution and uncertainty relation
7. Hamiltonian coordinates
8. Analogy mechanics – optics
Pupil stop defines:
1. chief ray angle $w$
2. aperture cone angle $u$

- The chief ray gives the center line of the oblique ray cone of an off-axis object point
- The coma rays limit the off-axis ray cone
- The marginal rays limit the axial ray cone
Definition of Field of View and Aperture

- Imaging on axis: circular / rotational symmetry
  Only spherical aberration and chromatical aberrations

- Finite field size, object point off-axis:
  - chief ray as reference
  - skew ray bundles: coma and distortion
  - Vignetting, cone of ray bundle not circular symmetric
  - to distinguish: tangential and sagittal plane
Quantitative measures of relative opening / size of accepted light cone

- Numerical aperture

\[ NA' = n' \cdot \sin u' \]

- F-number

\[ F\# = \frac{f'}{D_{Exp}} \]

- Approximation for small apertures:

\[ F\# = \frac{1}{2 \cdot NA'} \]
- Meridional rays:
  in main cross section plane

- Sagittal rays:
  perpendicular to main cross section plane

- Coma rays:
  Going through field point and edge of pupil

- Oblique rays:
  without symmetry
Tangential- and Sagittal Plane

- Off-axis object point:
  1. Meridional plane / tangential plane / main cross section plane contains object point and optical axis
  2. Sagittal plane: perpendicular to meridional plane through object point
Ray Fan and Ray Cone

- Ray fan: 2-dimensional plane set of rays
- Ray cone: 3-dimensional filled ray cone
- Transverse aberrations:
  Ray deviation from ideal image point in meridional and sagittal plane respectively
- The sampling of the pupil is only filled in two perpendicular directions along the axes
- No information on the performance of rays in the quadrants of the pupil
Pupil sampling for calculation of tranverse aberrations: all rays from one object point to all pupil points on x- and y-axis

- Two planes with 1-dimensional ray fans
- No complete information: no skew rays
Pupil Sampling

- Ray plots
- Spot diagrams

![Sagittal ray fan](image)

![Tangential ray fan](image)

![Whole pupil area](image)

![Tangential aberration](image)

![Sagittal aberration](image)
Pupil sampling in 3D for spot diagram:
all rays from one object point through all pupil points in 2D

Light cone completely filled with rays
Pupil Sampling

- Polar grid
- Cartesian
- Isoenergetic circular
- Hexagonal
- Statistical
- Artefacts due to regular gridding of the pupil of the spot in the image plane
- In reality a smooth density of the spot is true
- The line structures are discretization effects of the sampling
- The physical stop defines the aperture cone angle $u$

- The real system may be complex

- The entrance pupil fixes the acceptance cone in the object space

- The exit pupil fixes the acceptance cone in the image space

Ref: Julie Bentley
Properties of the Pupil

Relevance of the system pupil:

- Brightness of the image
  Transfer of energy

- Resolution of details
  Information transfer

- Image quality
  Aberrations due to aperture

- Image perspective
  Perception of depth

- Compound systems:
  matching of pupils is necessary, location and size
Entrance and Exit Pupil

- Entrance pupil
- Exit pupil
- Upper marginal ray
- Chief ray
- Lower coma ray
- Upper coma ray
- Lower marginal ray
- Outer field point of object
- Object point on axis
- Field point of image
- On axis point of image
- Stop
Position of the stop determines the path of the chief ray.

Quantitative parameter for the description:
parameter of eccentricity, relative position between pupil and field:
Pupil: \( h_{CR} = 0 \): \( \chi = -1 \)
Image: \( h_{MR} = 0 \): \( \chi = +1 \)

\[
\chi = \frac{h_{CR} - h_{MR}}{h_{CR} + h_{MR}}
\]
Pupil Mismatch

- Telescopic observation with different f-numbers
- Bad match of pupil location: key hole effect

\[
\begin{align*}
F# &= 2.8 \\
F# &= 8 \\
F# &= 22
\end{align*}
\]

Ref: H. Schlemmer
Vignetting

- Artificial vignetting:
  Truncation of the free area of the aperture light cone

- Natural Vignetting:
  Decrease of brightness according to $\cos w^4$ due to oblique projection of areas and changed photometric distances
- 3D-effects due to vignetting
- Truncation of the at different surfaces for the upper and the lower part of the cone
Vignetting

- Truncation of the light cone with asymmetric ray path for off-axis field points
- Intensity decrease towards the edge of the image
- Definition of the chief ray: ray through energetic centroid
- Vignetting can be used to avoid uncorrectable coma aberrations in the outer field
- Effective free area with extrem aspect ratio: anamorphic resolution
Vignetting

- Illumination fall off in the image due to vignetting at the field boundary
Helmholtz-Lagrange Invariant

- Product of field size $y$ and numerical aperture is invariant in a paraxial system
- Derivation at a single refracting surface:
  1. Common height $h$:
  
  
  $L = n \cdot y \cdot u = n' \cdot y' \cdot u'$

  $h = s \cdot u = s' \cdot u'$

  $w = \frac{y}{s}$, \hspace{1em} $w' = \frac{y'}{s'}$

  $nw = n'w'$

- The invariance corresponds to:
  1. Energy conservation
  2. Liouville theorem
  3. Invariant phase space volume (area)
  4. Constant transfer of information
Helmholtz-Lagrange Invariant

- Basic formulation of the Lagrange invariant:
  Uses image height, only valid in field planes

- General expression:
  1. Triangle SPB
     \[ w' = \frac{y_{CR}}{s'_{Exp}} \]
  2. Triangle ABO'
     \[ y'_{CR} = w' \left( s' - s'_{Exp} \right) \]
  3. Triangle SQA
     \[ u' = \frac{y_{MR}}{s'} \]
  4. Gives
     \[ L = n' u' y'_{CR} = n' \frac{y_{MR}}{s'} w' \left( s' - s'_{Exp} \right) = n' \left( y_{MR} w' - u' w' s'_{Exp} \right) \]
  5. Final result for arbitrary z:
     \[ L = n' \left[ w' y_{MR}(z) - u' y_{CR}(z) \right] \]
Optical Image formation:

- Sequence of pupil and image planes
- Matching of location and size of image planes necessary (trivial)
- Matching of location and size of pupils necessary for invariance of energy density
- In microscopy known as Köhler illumination
1. Slit diffraction
   Diffraction angle inverse to slit width D
   \[ \theta = \frac{\lambda}{D} \]

2. Gaussian beam
   Constant product of waist size \( w_o \) and divergence angle \( \theta_o \)
   \[ w_o \theta_o = \frac{\lambda}{\pi} \]
Laser optics: beam parameter product
waist radius times far field divergence angle

Minimum value of L:
TEM\textsubscript{oo} - fundamental mode

Elementary area of phase space:
Uncertainty relation in optics

Laser modes: discrete structure of phase space

Geometrical optics: quasi continuum

L is a measure of quality of a beam
small L corresponds to a good focussability

\[ L_{GB} = w_o \cdot \theta_o \]

\[ L_{GB} = \frac{\lambda}{\pi} \]

\[ L_{GB} = w_n \cdot \theta_n = \frac{\lambda}{\pi} \cdot (2n+1) \]
Phase Space: 90° - Rotation

- Transition pupil-image plane: 90° rotation in phase space
- Planes Fourier inverse
- Marginal ray: space coordinate $x$ --> angle $\theta'$
- Chief ray: angle $\theta$ --> space coordinate $x'$
Direct phase space representation of raytrace: spatial coordinate vs angle
Phase Space
Phase Space

Direct phase space representation of raytrace: spatial coordinate vs angle
Grin lens with aberrations in phase space:
- continuous bended curves
- aberrations seen as nonlinear angle or spatial deviations
Transfer of Energy in Optical Systems

- Conservation of energy
  \[ d^2 \Phi = d^2 \Phi' \]
- Differential flux (L: radiance)
  \[ d^2 \Phi = L \cdot \sin u \cdot \cos u \cdot dA \cdot du \cdot d\varphi \]
- No absorption
  \[ T = 1 \]
- Sine condition
  \[ ny \cdot \sin u = n' y' \cdot \sin u' \]
- Geometrical optic: Etendue, light gathering capacity
- Paraxial optic: invariant of Lagrange / Helmholtz
- General case: 2D
- Invariance corresponds to conservation of energy
- Interpretation in phase space: constant area, only shape is changed at the transfer through an optical system

\[
L_{Geo} = \frac{D_{field}}{2} \cdot n \cdot \sin u
\]

\[
L = n \cdot y \cdot u = n' \cdot y' \cdot u'
\]
Conservation of Energy

• Invariance of Energy:
  - constant area in phase space in the geometrical model
  - constant integral over density in the wave optical model

• Incoherent ensemble, quasi continuum:
  Jacobian matrix of transformation relates the coordinate changes

\[ J = \left( \frac{\partial L}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial L}{\partial y} \cdot \frac{\partial u}{\partial x} \right) \]

\[ \Delta x \Delta u = \text{const.} \]
Example Phase Space of an Array

Simplified pictures to the changes of the phase space density.
Etendue is enlarged, but no complete filling.

1) before array
2) separation of subapertures
3) lens effect of subapertures
4) in focal plane
5) in 2f-plane
6) far away
Generalization of paraxial picture:
Principal surface works as effective location of ray bending for object points near the optical axis (isoplanatic patch)

Paraxial approximation: plane
Can be used for all rays to find the imaged ray

Real systems with corrected sine-condition (aplanatic): principal sphere

The pupil sphere can not be used to construct arbitrary ray paths

If the sine correction is not fulfilled: more complicated shape of the artificial surface, that represents the ray bending
Pupil Sphere

- Pupil sphere: equidistant sine-sampling
Aplanatic system:

- Sine condition fulfilled
- Pupil has spherical shape
- Normalized canonical coordinates for pupil and field

\[
\begin{align*}
\bar{y}_p &= \frac{y_p}{h_{EnP}} \\
\bar{y}'_p &= \frac{y'_p}{h'_{Exp}} \\
\bar{y} &= \frac{n \cdot \sin u}{\lambda} \cdot y \\
\bar{y}' &= \frac{n' \cdot \sin u'}{\lambda} \cdot y'
\end{align*}
\]
Canonical Coordinates

- Normalized pupil coordinates
  \[ \bar{x}_p = \frac{x_p}{h_{EnP}} \quad \bar{x}'_p = \frac{x'_p}{h'_{ExP}} \]
  \[ \bar{y}_p = \frac{y_p}{h_{EnP}} \quad \bar{y}'_p = \frac{y'_p}{h'_{ExP}} \]

- Special case: aplanatic imaging
  \[ \bar{x}'_p = \bar{x}_p \]
  \[ \bar{y}'_p = \bar{y}_p \]

- Normalized field coordinates
  \[ \bar{x} = \frac{n \cdot \sin u_{sag}}{\lambda} \cdot x \quad \bar{x}' = \frac{n' \cdot \sin u'_{sag}}{\lambda} \cdot x' \]
  \[ \bar{y} = \frac{n \cdot \sin u_{\tan}}{\lambda} \cdot y \quad \bar{y}' = \frac{n' \cdot \sin u'_{\tan}}{\lambda} \cdot y' \]

- Special case: paraxial imaging
  \[ \bar{x}' = \bar{x} \]
  \[ \bar{y}' = \bar{y} \]

- Reference on chief ray
  \[ NA = n \cdot \sin u = n \cdot (\sin u_{TCO} - \sin u_{CR}) \]

- Reduced image side coordinates
  \[ \bar{x} = \frac{n \cdot \sin u_{sag}}{\lambda} \cdot x \]
  \[ \bar{y} = \frac{n \cdot (\sin u_{CR} - \sin u_{\tan})}{\lambda} \cdot y \]
## Analogy Optics - Mechanics

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<td>x</td>
<td>x</td>
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<tr>
<td>Impuls variable</td>
<td>p=mv</td>
<td>angle u, direction cosine p,</td>
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<td></td>
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<td>spatial frequency s_x, k_x</td>
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<td>equation of motion spatial</td>
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