Imaging and Aberration Theory

Lecture 1: Paraxial imaging
2013-10-17
Herbert Gross

Winter term 2013
Overview

- Time: Thursday, 14.00 – 15.30
- Location: Abbeanum, HS 2, Fröbelstieg 1
- Web page on IAP homepage under 'learning/materials' provides slides, exercises, solutions, informations
- Seminar: Exercises and solutions of given problems
  - time: Friday, 14.00 -15.30
  - Bespr.Raum 102 Abbeanum
  - starting date: 2012-10-25
- Shift of some dates could be possible
- Written examination, 90'
Literature

## Preliminary time schedule

<table>
<thead>
<tr>
<th>No.</th>
<th>Date</th>
<th>Topic</th>
<th>Details</th>
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<tbody>
<tr>
<td>1</td>
<td>17.10</td>
<td>Paraxial imaging</td>
<td>paraxial optics, fundamental laws of geometrical imaging, compound systems</td>
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<tr>
<td>2</td>
<td>24.10</td>
<td>Pupils, Fourier optics, Hamiltonian coordinates</td>
<td>pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates</td>
</tr>
<tr>
<td>3</td>
<td>31.10</td>
<td>Eikonal</td>
<td>Fermat Principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media</td>
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<tr>
<td>4</td>
<td>07.11</td>
<td>Aberration expansion</td>
<td>single surface, general Taylor expansion, representations, various orders, stop shift formulas</td>
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<tr>
<td>5</td>
<td>14.11</td>
<td>Representations of aberrations</td>
<td>different types of representations, fields of application, limitations and pitfalls, measurement of aberrations</td>
</tr>
<tr>
<td>6</td>
<td>21.11</td>
<td>Spherical aberration</td>
<td>phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders</td>
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<tr>
<td>7</td>
<td>28.11</td>
<td>Distortion and coma</td>
<td>phenomenology, relation to sine condition, aplanatic sytems, effect of stop position, various topics, correction options</td>
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<td>8</td>
<td>05.12</td>
<td>Astigmatism and curvature</td>
<td>phenomenology, Coddington equations, Petzval law, correction options</td>
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<tr>
<td>9</td>
<td>12.12</td>
<td>Chromatical aberrations</td>
<td>Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary specotrum</td>
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<tr>
<td>10</td>
<td>19.12</td>
<td>Further reading on aberrations</td>
<td>sensitivity in 3rd order, structure of a system, analysis of optical systems, lens contributions, Sine condition, isoplanatism, sine condition, Herschel condition, relation to coma and shift invariance, pupil aberrations, relation to Fourier optics</td>
</tr>
<tr>
<td>11</td>
<td>09.01</td>
<td>Wave aberrations</td>
<td>definition, various expansion forms, propagation of wave aberrations, relation to PSF and OTF</td>
</tr>
<tr>
<td>12</td>
<td>16.01</td>
<td>Zernike polynomials</td>
<td>special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, recalculation for offset, ellipticity, measurement</td>
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<tr>
<td>13</td>
<td>23.01</td>
<td>Miscellaneous</td>
<td>Intrinsic and induced aberrations, Aldi theorem, telecentric case, afocal case, aberration balancing, Scheimpflug imaging, Fresnel lenses, statistical aberrations</td>
</tr>
<tr>
<td>14</td>
<td>30.01</td>
<td>Vectorial aberrations</td>
<td>Introduction, special cases, actual research, anamorphotic, partial symmetric</td>
</tr>
<tr>
<td>15</td>
<td>06.02</td>
<td></td>
<td></td>
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</tbody>
</table>
1. Cardinal elements
2. Lens properties
3. Imaging, magnification
4. Afocal systems and telecentricity
5. Paraxial approximation
6. Matrix calculus
Principal purpose of calculations:
- System, data of the structure (radii, distances, indices, ...)
- Function, data of properties, quality performance (spot diameter, MTF, Strehl ratio, ...)

Imaging model with levels of refinement:
- Paraxial model (focal length, magnification, aperture, ...)
- Analytical approximation and classification (aberrations, ...)
- Geometrical optics (transverse aberrations, wave aberration, distortion, ...)
- Wave optics (point spread function, OTF, ...)

Ref: W. Richter
Single Surface

- Single surface between two media
  Radius \( r \), refractive indices \( n, n' \)
- Imaging condition, paraxial
- Abbe invariant
  alternative representation of the imaging equation

\[
\frac{n'}{s'} - \frac{n}{s} = \frac{n'-n}{r} = \frac{1}{f'}
\]

\[
Q_s = n \cdot \left( \frac{1}{r} - \frac{1}{s} \right) = n' \cdot \left( \frac{1}{r} - \frac{1}{s'} \right)
\]
Cardinal elements of a lens

- **Focal points:**
  1. incoming parallel ray intersects the axis in $F'$
  2. ray through $F$ is leaves the lens parallel to the axis

- **Principal plane $P$:**
  location of apparent ray bending

- **Nodal points:**
  Ray through $N$ goes through $N'$ and preserves the direction
Notations of a lens

- **P** principal point
- **S** vertex of the surface
- **F** focal point
- **s** intersection point of a ray with axis
- **f** focal length $PF$
- **r** radius of surface curvature
- **d** thickness $SS'$
- **n** refractive index
Main properties of a lens

- Main notations and properties of a lens:
  - radii of curvature $r_1$, $r_2$
  - curvatures $c$
  - sign: $r > 0$: center of curvature is located on the right side
  - thickness $d$ along the axis
  - diameter $D$
  - index of refraction of lens material $n$

- Focal length (paraxial)
  \[
  f = \frac{y}{\tan u}, \quad f' = \frac{y'}{\tan u'}
  \]

- Optical power
  \[
  F = -\frac{n}{f} = \frac{n'}{f'}
  \]

- Back focal length intersection length, measured from the vertex point
  \[
  s_{F'} = f' + s_{P'}
  \]
Different shapes of singlet lenses:
1. bi-, symmetric
2. plane convex / concave, one surface plane
3. Meniscus, both surface radii with the same sign

Convex: bending outside
Concave: hollow surface

Principal planes \( P, P' \): outside for meniscus shaped lenses
- Ray path at a lens of constant focal length and different bending

- The ray angle inside the lens changes

- The ray incidence angles at the surfaces changes strongly

- The principal planes move
  For invariant location of P, P' the position of the lens moves
Magnification parameter $M$:
defines ray path through the lens

$$M = \frac{U' + U}{U' - U} = \frac{1 + m}{1 - m} = \frac{2f}{s} + 1 = \frac{2f}{s'} - 1$$

- Special cases:
  1. $M = 0$: symmetrical 4f-imaging setup
  2. $M = -1$: object in front focal plane
  3. $M = +1$: object in infinity

- The parameter $M$ strongly influences the aberrations
- Optical Image formation:
  All ray emerging from one object point meet in the perfect image point

- Region near axis:
  Gaussian imaging
  Ideal, paraxial

- Image field size:
  Chief ray

- Aperture/size of light cone:
  Marginal ray defined by pupil stop
Formulas for surface and lens imaging

- Single surface imaging equation:
  \[ \frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{r} = \frac{1}{f'} \]

- Thin lens in air focal length:
  \[ \frac{1}{f'} = (n - 1) \cdot \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]

- Thin lens in air with one plane surface, focal length:
  \[ f' = \frac{r}{n - 1} \]

- Thin symmetrical bi-lens:
  \[ f' = \frac{r}{2 \cdot (n - 1)} \]

- Thick lens in air focal length:
  \[ \frac{1}{f'} = (n - 1) \cdot \left( \frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{(n - 1)^2 d}{n \cdot r_1 r_2} \]
Imaging by a lens in air: lens makers formula

\[
\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}
\]

Magnification

\[
m = \frac{s'}{s}
\]

Real imaging:

\( s < 0, s' > 0 \)

Intersection lengths \( s, s' \) measured with respective to the principal planes \( P, P' \)
Imaging by a Lens

- Ranges of imaging
  - Location of the image for a single lens system
- Change of object location
- Image could be:
  1. real / virtual
  2. enlarged/reduced
  3. in finite/infinite distance
- Imaging equation according to Newton: distances $z$, $z'$ measured relative to the focal points 

$$z \cdot z' = f \cdot f'$$
Multi-Surface Systems

- Two lenses with distance $d$
  \[ F = F_1 + F_2 - \frac{d \cdot F_1 \cdot F_2}{n} \]

- Focal length
  distance of inner focal points $e$
  \[ f = \frac{f_1 \cdot f_2}{f_1 + f_2 - d} = \frac{f_1 \cdot f_2}{e} \]

- Sequence of thin lenses close together
  \[ F = \sum_k F_k \]

- Sequence of surfaces with relative ray heights $h_j$, paraxial
  \[ F = \sum_k \frac{h_k}{h_1} \cdot (n'_k - n_k) \cdot \frac{1}{r_k} \]

- Magnification
  \[ m = \frac{s'_1 \cdot s'_2 \cdot \ldots \cdot s'_k}{s_1 \cdot s_2 \cdot \ldots \cdot s_k} \cdot \frac{n_1}{n'_k} \]
Two-Lens System

- Focal length
- e: tube length
- Image location

\[ f' = \frac{f'_1 \cdot f'_2}{f'_1 + f'_2 - d} = \frac{f'_1 \cdot f'_2}{e} \]

\[ s'_2 = \frac{(f'_1 - d) \cdot f'_2}{f'_1 + f'_2 - d} = \frac{(f'_1 - d) \cdot f'}{f'_1} \]
Magnification

- Lateral magnification for finite imaging
- Scaling of image size

\[ m = \frac{y'}{y} = -\frac{f \cdot \tan u}{f' \cdot \tan u'} \]
Angle Magnification

- Afocal systems with object/image in infinity
- Definition with field angle $w$
  angular magnification

\[ \gamma = \frac{\tan w'}{\tan w} = \frac{nh}{n'h'} \]

- Relation with finite-distance magnification

\[ m \cdot \gamma = -\frac{f}{f'} \]
Axial magnification

Approximation for small $\Delta z$ and $n = n'$
Definition of Field of View and Aperture

- Imaging on axis: circular / rotational symmetry
  Only spherical aberration and chromatical aberrations

- Finite field size, object point off-axis:
  - chief ray as reference
  - skew ray bundles: coma and distortion
  - Vignetting, cone of ray bundle not circular symmetric
  - to distinguish: tangential and sagittal plane
The Special Infinity Cases

- Simple case:
  - object, image and pupils are lying in a finite distance
  - non-telecentric relay systems

- Special case 1:
  - object at infinity
  - object sided afocal
  - example: camera lens for distant objects

- Special case 2:
  - image at infinity
  - image sided afocal
  - example: eyepiece

- Special case 3:
  - entrance pupil at infinity
  - object sides telecentric
  - example: camera lens for metrology

- Special case 4:
  - exit pupil at infinity
  - image sided telecentric
  - example: old fashion lithographic lens
The Special Infinity Cases

- Very special: combination of above cases
  Examples:
  - both sided telecentric: 4f-system, lithographic lens
  - both sided afocal: afocal zoom
  - object sided telecentric, image sided afocal: microscopic lens

- Notice: telecentricity and afocality can not be combined on the same side of a system
**Telecentricity**

- **Special stop positions:**
  1. stop in back focal plane: object sided telecentricity
  2. stop in front focal plane: image sided telecentricity
  3. stop in intermediate focal plane: both-sided telecentricity

- **Telecentricity:**
  1. pupil in infinity
  2. chief ray parallel to the optical axis
Telecentricity

- Double telecentric system: stop in intermediate focus
- Realization in lithographic projection systems
Paraxial Approximation

- Paraxiality is given for small angles relative to the optical axis for all rays.
- Large numerical aperture angle $u$ violates the paraxiality, spherical aberration occurs.
- Large field angles $w$ violates the paraxiality, coma, astigmatism, distortion, field curvature occurs.
- Classification of systems with field and aperture size
- Scheme is related to size, correction goals and etendue of the systems
- Aperture dominated: Disk lenses, microscopy, Collimator
- Field dominated: Projection lenses, camera lenses, Photographic lenses
- Spectral width as a correction requirement is missed in this chart
- Incidence angles for chief and marginal ray
- Aperture dominant system
- Primary problem is to correct spherical aberration
Photographic lens

- Incidence angles for chief and marginal ray
- Field dominant system
- Primary goal is to control and correct field related aberrations: coma, astigmatism, field curvature, lateral color
Paraxial approximation:

- Small angles of rays at every surface
- Small incidence angles allows for a linearization of the law of refraction $n \cdot i = n' \cdot i'$
- All optical imaging conditions become linear (Gaussian optics), calculation with ABCD matrix calculus is possible
- No aberrations occur in optical systems
- There are no truncation effects due to transverse finite sized components
- Serves as a reference for ideal system conditions
- Is the fundament for many system properties (focal length, principal plane, magnification,...)
- The sag of optical surfaces (difference in $z$ between vertex plane and real surface intersection point) can be neglected
- All waves are plane of spherical (parabolic)
- The phase factor of spherical waves is quadratic

$$E(x) = E_0 \cdot e^{-\frac{i \pi x^2}{\lambda R}}$$
Paraxial approximation

- Law of refraction for finite angles $I, I'$
  \[ n \cdot \sin I = n' \cdot \sin I' \]

- Taylor expansion
  \[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]

- Linear formulation of the law of refraction for small angles $i, i'$
  \[ n \cdot i = n' \cdot i' \]

- Relative direction error of the paraxial approximation
  \[ \varepsilon = \frac{i' - I'}{I'} = \frac{n \cdot i}{n'} \arcsin \left( \frac{n \cdot \sin i}{n'} \right) - 1 \]
Linear Collineation

- General transform object \( \rightarrow \) image space
  \[ x' = F(x, y, z), \quad y' = F(x, y, z), \quad z' = F(x, y, z) \]

- General rational transformation with linear expression
  \[ x' = \frac{F_1}{F_0}, \quad y' = \frac{F_2}{F_0}, \quad z' = \frac{F_3}{F_0} \]

- Describes linear collinear transform \( x,y,z \rightarrow x',y',z' \)

- Inversion
  \[ x = \frac{F_1'}{F_0'}, \quad y = \frac{F_2'}{F_0'}, \quad z = \frac{F_3'}{F_0'} \]

- Analog in the image space
  \[ F'_j = a_j x + b_j y + c_j z + d_j, \quad j = 0, 1, 2, 3 \]

- Inserted in only 2 dimensions

- Focal lengths
  from conditions \( F_0 = 0 \) and \( F_0' = 0 \)

- Principal planes
- Finite angles: \( \tan(u) \) must be taken:

Magnification:

\[
m = \frac{\tan u'}{\tan u}
\]

Focal length:

\[
\frac{1}{f'} = \frac{\tan u' - \tan u}{h}
\]

Invariant:

\[
y \tan u = n'y' \tan u'
\]
Matrix Calculus

- Paraxial raytrace transfer
  \[ y_j = y_{j-1} + d_{j-1} \cdot U_{j-1} \]
  \[ U_j' = U_{j-1} \]

- Matrix formulation
  \[
  \begin{pmatrix}
  y'_j \\
  U'_j
  \end{pmatrix} =
  \begin{pmatrix}
  1 & d_{j-1} \\
  0 & 1
  \end{pmatrix}
  \begin{pmatrix}
  y_j \\
  U_j
  \end{pmatrix}
  \]

- Matrix formalism for finite angles
  \[
  \begin{pmatrix}
  y'_j \\
  \tan u'_j
  \end{pmatrix} =
  \begin{pmatrix}
  A & B \\
  C & D
  \end{pmatrix}
  \begin{pmatrix}
  y_j \\
  \tan u_j
  \end{pmatrix}
  \]

- Paraxial raytrace refraction
  \[ y_j = y_{j-1} \]
  \[ i_j = \rho_j \cdot y_j + U_{j-1} \]
  \[ i_j' = \frac{n_j}{n_j'} i_j \]
  \[ U_j' = U_{j-1} - i_j + i_j' \]

- Inserted
  \[ U_j' = -\rho_j \cdot \frac{(n_j' - n_j)}{n_j} y_j + \frac{n_j}{n_j'} U_{j-1} \]

- Matrix formulation
  \[
  \begin{pmatrix}
  y'_j \\
  U'_j
  \end{pmatrix} =
  \begin{pmatrix}
  1 & -\rho_j \cdot \frac{(n_j' - n_j)}{n_j} \\
  n_j & \frac{n_j}{n_j'}
  \end{pmatrix}
  \begin{pmatrix}
  y_j \\
  U_j
  \end{pmatrix}
  \]
Matrix Formulation of Paraxial Optics

- Linear relation of ray transport
- Simple case: free space propagation
- Advantages of matrix calculus:
  1. simple calculation of component combinations
  2. Automatic correct signs of properties
  3. Easy to implement
- General case:
  paraxial segment with matrix ABCD-matrix:

\[
\begin{pmatrix}
  x' \\
  u'
\end{pmatrix}
= \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  x \\
  u
\end{pmatrix}
= M \begin{pmatrix}
  x \\
  u
\end{pmatrix}
\]
- Linear transfer of spatial coordinate $x$ and angle $u$
- Matrix representation
- Lateral magnification for $u=0$
- Angle magnification of conjugated planes
- Refractive power for $u=0$
- Composition of systems
- Determinant, only 3 variables

\[ x' = Ax + Bu \]
\[ u' = Cx + Du \]

\[
\begin{pmatrix} x' \\ u' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = M \begin{pmatrix} x \\ u \end{pmatrix}
\]

\[ A = x'/x = \beta \]
\[ D = u'/u = \gamma \]
\[ C = u'/x \]

\[
M = M_k \cdot M_{k-1} \cdots M_2 \cdot M_1
\]

\[
\det M = AD - BC = \frac{n}{n'}
\]
Matrix Formulation of Paraxial Optics

- System inversion
  \[ M^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \]

- Transition over distance L
  \[ M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \]

- Thin lens with focal length f
  \[ M = \begin{pmatrix} \frac{1}{f} & 0 \\ -1 & 1 \end{pmatrix} \]

- Dielectric plane interface
  \[ M = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{pmatrix} \]

- Afocal telescope
  \[ M = \begin{pmatrix} 1 & L \\ \frac{1}{\Gamma} & \Gamma \end{pmatrix} \]
Matrix Formulation of Paraxial Optics

- Calculation of intersection length
  \[ s' = \frac{A \cdot s + B}{C \cdot s + D} \]

- Magnifications:
  1. lateral
    \[ \beta = \frac{AD - BC}{C \cdot s + D} \]
  2. angle
    \[ \gamma = C \cdot s + D = \frac{AD - BC}{A - C \cdot s'} \]
  3. axial, depth
    \[ \alpha = \frac{ds'}{ds} = \frac{AD - BC}{(C \cdot s + D)^2} \]

- Principal planes
  \[ a_H = \frac{AD - BC - D}{C} \quad a_{H'} = \frac{A - 1}{C} \]

- Focal points
  \[ a_F = \frac{A}{C} \quad a_F = -\frac{D}{C} \]
Decomposition of ABCD-Matrix

- 2x2 ABCD-matrix of a system in air: 3 arbitrary parameters
- Every arbitrary ABCD-setup can be decomposed into a simple system
- Decomposition in 3 elementary partitions is always possible

- Case 1: C ≠ 0
  one lens, 2 transitions

\[
 M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix}
\]

- System data

\[
 L_2 = \frac{A - 1}{C}
\]
\[
 f = -\frac{1}{C}
\]
\[
 L_1 = \frac{D - 1}{C}
\]
Decomposition of ABCD-Matrix

- Case 2: B ≠ 0
two lenses, one transition

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -1/f_2 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1/f_1 & 0 \\ 0 & 1 \end{pmatrix}
\]

- System data:

\[
f_1 = -\frac{B}{A-1}
\]
\[
L = B
\]
\[
f_2 = -\frac{B}{D-1}
\]