Imaging and Aberration Theory

Lecture 13: Miscellaneous
2013-02-01
Herbert Gross
<table>
<thead>
<tr>
<th>No.</th>
<th>Date</th>
<th>Topic</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19.10.</td>
<td>Preliminary time schedule</td>
<td>paraxial imaging, paraxial optics, fundamental laws of geometrical imaging, compound systems</td>
</tr>
<tr>
<td>2</td>
<td>26.10.</td>
<td>Paraxial imaging</td>
<td>pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates</td>
</tr>
<tr>
<td>3</td>
<td>02.11.</td>
<td>Eikonal</td>
<td>Fermat Principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media</td>
</tr>
<tr>
<td>4</td>
<td>09.11.</td>
<td>Aberration expansion</td>
<td>single surface, general Taylor expansion, representations, various orders, stop shift formulas</td>
</tr>
<tr>
<td>5</td>
<td>16.11.</td>
<td>Representations of aberrations</td>
<td>different types of representations, fields of application, limitations and pitfalls, measurement of aberrations</td>
</tr>
<tr>
<td>6</td>
<td>23.11.</td>
<td>Spherical aberration</td>
<td>phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders</td>
</tr>
<tr>
<td>7</td>
<td>07.12.</td>
<td>Distortion and coma</td>
<td>phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options</td>
</tr>
<tr>
<td>8</td>
<td>14.12.</td>
<td>Astigmatism and curvature</td>
<td>phenomenology, Coddington equations, Petzval law, correction options</td>
</tr>
<tr>
<td>9</td>
<td>21.12.</td>
<td>Chromatical aberrations</td>
<td>Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum</td>
</tr>
<tr>
<td>10</td>
<td>11.01.</td>
<td>Further reading on aberrations</td>
<td>sensitivity in 3rd order, structure of a system, analysis of optical systems, lens contributions, Sine condition, isoplanatism, sine condition, Herschel condition, relation to coma and shift invariance, pupil aberrations, relation to Fourier optics and phase space</td>
</tr>
<tr>
<td>11</td>
<td>18.01.</td>
<td>Wave aberrations</td>
<td>definition, various expansion forms, propagation of wave aberrations, relation to PSF and OTF</td>
</tr>
<tr>
<td>12</td>
<td>25.01.</td>
<td>Zernike polynomials</td>
<td>special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, recalculation for offset, ellipticity, measurement</td>
</tr>
<tr>
<td>13</td>
<td>01.02.</td>
<td>Miscellaneous</td>
<td>Intrinsic and induced aberrations, Aldi theorem, telecentric case, afocal case, aberration balancing, Delano diagram, Scheimpflug imaging, Fresnel lenses, statistical aberrations</td>
</tr>
<tr>
<td>14</td>
<td>08.02.</td>
<td>Vectorial aberrations</td>
<td>Introduction, special cases, actual research, anamorphic, partial symmetric</td>
</tr>
</tbody>
</table>
1. Telecentric systems
2. Scheimpflug imaging
3. Diffractive components
4. Fresnel lenses
5. Aldi theorem
6. Induced aberrations
7. Caustics
8. Illumination and aberrations
Telecentricity

- Imaging with / without telecentricity
- Magnification as function of distance
- Size of optical system

Ref: W. Osten
Telecentricity

- Special stop positions:
  1. stop in back focal plane: object sided telecentricity
  2. stop in front focal plane: image sided telecentricity
  3. stop in intermediate focal plane: both-sided telecentricity

- Telecentricity:
  1. pupil in infinity
  2. chief ray parallel to the optical axis

- Problem in practical systems:
  large diameters necessary

\[ D > 2 \cdot (y_{\text{max}} + f \cdot NA) \]
Telecentricity

- Double telecentric system: stop in intermediate focus
- Realization in lithographic projection systems
Example system
Problem: coma and vignetting disturb telecentricity
Definition of telecentricity deviation:
range of telecentricity for accepted lateral deviation $\Delta y'$ for finite chief ray angle $\omega$

$$\Delta s = \frac{\Delta y'}{m \cdot \tan \omega}$$

- 480 nm
- 587 nm
- 656 nm
Scheimpflug Imaging

- Imaging with tilted object plane
- If principal plane, object and image plane meet in a common point: Scheimpflug condition, sharp imaging possible
- Scheimpflug equation

\[
\frac{s}{s'} = \frac{\tan \theta - \tan \varphi}{\tan \theta' - \tan \varphi}
\]
Scheimpflug Imaging

- More general case of finite locations of the principal planes
- Derivation of Scheimpflug imaging condition with depth magnification

\[ \alpha = \frac{z'}{z} = -m_o^2 = -\frac{y'^2}{y^2} \]
- General property:
  - Magnification depends on location in the object plane
  - anamorphotic magnification
  - corresponds to macroscopic keystone distortion

\[ V = m'_{\perp} \cdot \frac{\sin \theta \cdot \cos \varphi}{f'} \]

- Imaging Relation

\[ m_y' = \left( \frac{s \cdot m_{\perp}'}{s + y \cdot (1 - m_{\perp}') \cdot \sin \theta} \right)^2 \cdot \frac{\sin \theta}{\sin \theta'} \]

\[ m_x' = \frac{s \cdot m_{\perp}'}{s + y \cdot (1 - m_{\perp}') \cdot \sin \theta} \]

\[ \frac{\tan \theta'}{\tan \theta} = m_{\perp}' = \frac{s'}{s} \]
Scheimpflug Imaging

- Example for oblique imaging with Scheimpflug condition
- Sharp image
- Strong distortion: example: keystone distortion

Ref: W. Osten
Deviation of Light

Mechanisms of light deviation and ray bending

- Refraction
  \[ n \cdot \sin \theta = n' \cdot \sin \theta' \]
  \[ \theta = -\theta' \]
- Reflection
- Diffraction according to the grating equation
  \[ g \cdot (\sin \theta - \sin \theta_o) = m \cdot \lambda \]
- Scattering (non-deterministic)
- Original lens height profile $h(x)$
- Wrapping of the lens profile: $h_{\text{red}}(x)$ Reduction on maximal height $h_{2\pi}$
- Digitalization of the reduced profile: $h_q(x)$
Diffracting Surfaces

- Surface with grating structure:
  new ray direction follows the grating equation
- Local approximation in the case of space-varying grating width
  \[ \vec{s}' = \frac{n}{n'} \cdot \vec{s} + \frac{m \lambda \hat{g}}{n' d} \cdot \hat{g} + \gamma \cdot \vec{e} \]
- Raytrace only into one desired diffraction order
- Notations:
  \( g \) : unit vector perpendicular to grooves
  \( d \) : local grating width
  \( m \) : diffraction order
  \( e \) : unit normal vector of surface
- Applications:
  - diffractive elements
  - line gratings
  - holographic components
Diffractive Optics

- Local micro-structured surface

- Location of ray bending: macroscopic carrier surface

- Direction of ray bending: local grating micro-structure

- Additional degree of freedom:
  - independent determination of
    1. ray bending location (carrier surface)
    2. ray bending direction (local grating)

- First effect corresponds to asphere
  Second effect corresponds to plane grating
Sweatt Modell of DOEs

- Local description of arbitrary ray bending as general asphere with phase function $\Phi$
- Ray bending performed in a thin layer: large $n$ and small height $z^*$
  Typically: $n = 10^4$
- Equivalent between gradient of phase and local grating constant $g(x,y)$
  generalized grating equation
- Conventional raytracing algorithms can be used

$$\Phi(x, y) = k \cdot \sum_{j,m} c_{jm} \cdot x^j \cdot y^m$$
$$\Phi(x, y) = 2\pi \cdot n \cdot z(x, y) = 2\pi \cdot n^* \cdot z^*(x, y)$$
$$g(x, y) = \frac{2\pi \cdot q}{|\nabla \Phi(x, y)|}$$

![Diagram of Sweatt lens and equivalent refractive aspherical lens](image_url)
- Intensity of grating diffraction pattern (scalar approximation $g \gg \lambda$)

- Product of slit-diffraction and interference function

- Maxima of pattern: coincidence of peaks of both functions: grating equation

$$g \cdot (\sin \theta - \sin \theta_o) = m \cdot \lambda$$

- Angle spread of orders decrease with growing number of periods $N$

- Oblique phase gradient:
  - relative shift of both functions
  - selection of peaks/order
  - basic principle of blazing

$$I = N^2 \cdot g^2 \cdot \left[ \frac{\sin \left( \frac{\pi ug}{\lambda} \right)}{\left( \frac{\pi ug}{\lambda} \right)} \cdot \frac{\sin \left( \frac{N \pi ug}{\lambda} \right)}{N \cdot \sin \left( \frac{\pi u}{\lambda} \right)} \right]^2$$

$$u = \frac{\pi}{\lambda} \sin \theta$$
**Diffraction Orders**

- Usually all diffraction orders are obtained simultaneously.
- Blazed structure: suppression of perturbing orders
  - Only possible for one wavelength and one incident angle.
- Unwanted orders: false light, contrast and efficiency reduced.
Lens with diffractive structured surface: hybrid lens

Refractive lens: dispersion with Abbe number $\nu = 25...90$

Diffractive lens: equivalent Abbe number

Combination of refractive and diffractive surfaces: achromatic correction for compensated dispersion

Usually remains a residual high secondary spectrum

Broadband color correction is possible but complicated

\[ \nu_d = \frac{\lambda_d}{\lambda_F - \lambda_C} = -3.453 \]
Color Correction of a Hybrid System

- Principle of achromatic correction
- Ratio of Abbe numbers defines refractive power distribution
- Diffractive element: Abbe number $\nu = -3.45$

\[
F_{\text{refr}} = F \cdot \frac{V_{\text{g}}}{V_{\text{g}} - V_{\text{d}}}
\]

\[
F_{\text{refr}} = -F \cdot \frac{V_{\text{d}}}{V_{\text{g}} - V_{\text{d}}}
\]

- Diffractive element gets only approx. 5% of the refractive power
- Dispersion by grating diffraction:
  Abbe number: small and negative!

Relative partial dispersion

Consequence:
Large secondary spectrum

- \( v_e = \frac{\lambda_e}{\lambda_{F'} - \lambda_{C'}} = -3.330 \)

- \( P_{g,F'} = \frac{\lambda_g - \lambda_{F'}}{\lambda_{F'} - \lambda_{C'}} = 0.2695 \)

\( n - P \)-diagram

Diffractive Optics: Dispersion
- Achromatic doublet
  Ratio of Abbe numbers and refractive powers
- Visible : 5% refractive power of DOE
  Infrared : strong variations depending on substrat material
Spherical Hybrid Achromate

- Classical achromate:
  - two lenses, different glasses
  - strong curved cemented surface

- Hybride achromate:
  - one lens
  - one surface spherical with diffractive structure
  - tolerances relaxed
- Types of achromates
  a) Spherical surfaces un cemented
  b) Hybrid with aspherical and diffractive surface
- Relaxed tolerances for hybrid solution
Primary Aberrations of a Diffractive Lens

- Expansion of the optical pathlength for one field point: Primary Seidel aberrations:

\[ W(r) = \frac{2\pi}{\lambda} \left( \frac{r^2}{2f} - \frac{r^4}{8f^3} + \frac{wr^3}{2f^2} - \frac{3w^2r^2}{4f} \right) \]

- No field curvature

- No distortion (stop at lens)

- Ray bending in a plane corresponds to linear collineation

- Equivalent bending of lens

\[ X = 2f \cdot c_{\text{diff}} = \frac{c_{\text{diff}}}{m\lambda \cdot a_2} \]
Optimization of a Hybrid Lens

- Seidel spherical aberration of a hybrid lens

\[ S_{sph} = S_{ref} + S_{dif} = \frac{y^4 F_{ref}^3}{4} \left[ \frac{n+2}{n(n-1)^2} + \left( \frac{n}{n-1} \right)^2 - \frac{4(n+1)}{n(n-1)} + \frac{3n+2}{n} \right] - 8m\lambda \cdot A_4 y^4 \]

- Optimal bending: choice of \( A_4 \)

\[ A_4 = \frac{F_{ref}^3}{32m\lambda} \left[ \frac{n+2}{n(n-1)^2} + \left( \frac{n}{n-1} \right)^2 - \frac{4(n+1)}{n(n-1)} + \frac{3n+2}{n} \right] \]

- Gaussian aberration large

\[ \Delta s_{gauss} = \frac{\Delta s_{ref,sph}}{\nu_{dif}} \]

- Optimization of refractive power of the DOE (10% instead of 5%)

\[ F_{dif} = F \cdot \left( \frac{\nu_{dif} - \nu_{ref}}{\nu_{dif} - \nu_{ref} \cdot \Delta s_{ref,sph}} \right) \]
\[ F_{ref} = -F \cdot \left( \frac{\nu_{ref}}{\nu_{dif} - \nu_{ref} \cdot \Delta s_{ref,sph}} \right) \]
- Combination of DOE and aspherical carrier substrate
- Full usage of degrees of freedom
Different approaches for broad band achromatic correction with DOE

Problems:
1. Efficiency
2. Secondary spectrum

Division of aperture into rings with blazing structures for different wavelengths
Two wavelength-design: bi-blaze
Structures complicated, efficiency low

### Broad Band Achromatization

<table>
<thead>
<tr>
<th></th>
<th>Hybrid refractive / diffractive</th>
<th>Higher order</th>
<th>Multi-material strategy</th>
<th>Bi-Blazing strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broad band efficiency</td>
<td>high</td>
<td>medium</td>
<td>high</td>
<td>medium</td>
</tr>
<tr>
<td>Minimal number of components</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Minimal number of materials</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Deep structures necessary</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>possible</td>
</tr>
<tr>
<td>Special effort for centering and adjustment</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

Division of aperture into rings with blazing structures for different wavelengths
Two wavelength-design: bi-blaze
Structures complicated, efficiency low

![Diagram](image)
- Large depth of relief: blaze-wavelengths numbers of orders: m, p

\[ \lambda_{\text{blaze}} = m \cdot \frac{\lambda_o}{n(K_o) - 1} \cdot \frac{n(\lambda_{\text{blaze}}) - 1}{p} \approx \frac{m \cdot \lambda_o}{p} \]

- Gives a smooth uniform broad band efficiency (scalar approximation \( g >> \lambda \))

![Graph showing efficiency \( \eta \) versus wavelength \( \lambda \).](image)
Diffractive Optics: Nearly Index Matching

- Nearly index matched materials
  special selection of materials necessary

\[ \frac{\lambda_e}{\lambda_{F'} - \lambda_C} \left( \frac{n_1(\lambda_e) - 1}{v_1} - \frac{n_2(\lambda_e) - 1}{v_2} \right) = n_1(\lambda_e) - n_2(\lambda_e) \]

- Height of second layer

\[ h = \frac{\lambda_e}{n_1(\lambda_e) - n_2(\lambda_e)} \]

- Efficiency

\[ \eta_{nim} = \text{sinc}^2 \left( \pi \cdot \frac{\lambda_o}{\lambda} \cdot \frac{n_1(\lambda) - n_2(\lambda)}{n_1(\lambda_o) - n_2(\lambda_o)} - \pi \cdot m \right) \]
- Double diffractive element sandwich
  - Achromatization by adapted compensation of materials for 2 wavelengths
  - Residual dispersion
  \[ \Phi = \frac{2\pi}{\lambda} \cdot \left\{ d_1 \cdot [n_1(\lambda) - n_g(\lambda)] + d_2 \cdot [n_2(\lambda) - n_g(\lambda)] \right\} \]

- Optimized depth of structures allows correction for 4 wavelengths

- Realized in Canon photographic lens manufacturing properties critical too expansive for consumer products
- Straylight suppression by proper DOE location and rear stop
Fresnel Lenses

- **Principle:**
  Aspherical surface with reduced sag

- **Advantages:**
  1. smaller weight
  2. length reduced
  3. shorter path inside glass
  4. multi functions possible by segmentation

- **Disadvantages:**
  1. imaging quality decreased
  2. manufacturing more complicated
  3. larger straylight
  4. structured illumination
Fresnel Lenses

- Slopes linear approximated
- Fresnel principle violated
- Macro- and micro imaging
- Total internal reflection limits maximum aperture

\[ \tan \phi = \frac{s'}{s} \]

\[ s' = 0.5 \, s \]
\[ s' = s \]
\[ s' = 3 \, s \]
\[ s' = >> s \]
Fresnel Lenses

- Aberrations:
  - defect of thickness
  - partly compensated by corrected slope
- Sine condition not fulfilled:
  large coma aberrations
- Linear slope cones: micro imaging
- Fresnel lenses: sine condition not fulfilled
  - principal plane, no sphere
  - focal length depends on ray height

$$f_{real} = \frac{f_{parax}}{\cos \theta}$$

- Large perturbation of homogeneous illumination

![Graph showing the relationship between L/L₀ and θ for different refractive indices n = 1.5 and n = 2.0.](image)
Aldis Theorem

- Aldis theorem: surface contribution of transverse aberration of all orders
- Calculation by tracing two rays:
  1. paraxial marginal ray
  2. finite ray

- **H**: Lagrange Invariant
  \[ H = n'_k u'_k y' \]

- **A**: Refraction invariant
  \[ A_j = n_j i_j = n_j (h_j c_j + u_j) \]

- Transverse aberrations

\[
\Delta x' = \frac{-1}{n'_k u'_k s'_k z_k} \sum_{j=1}^{k} \left[ A_j z_j \Delta s_{xj} + \frac{A_j x_j}{s'_z + s_{zj}} \Delta \left( s_{xj}^2 + s_{yj}^2 \right) \right] \\
\Delta y' = \frac{-1}{n'_k u'_k s'_k z_k} \sum_{j=1}^{k} \left[ A_j z_j \Delta s_{yj} + \frac{A_j y_j - H}{s'_z + s_{zj}} \Delta \left( s_{xj}^2 + s_{yj}^2 \right) \right]
\]
Aldis Theorem

- Advantage of Aldis theorem: contain all orders
- Larger differences for surfaces/cases with higher order contributions
- Usually, the reference is the paraxial ray, therefore distortion is taken into account
- A known formulation is available for aspherical surfaces in centered systems
- A specialized equation must be used for the case of image in infinity
- More general 3D geometries are not supported
- More general formulations are possible (Brewer)
- Disadvantage of Aldis theorem: only for one ray
Aldis Theorem

- Example Achromate
  - Seidel and Aldis contributions at every surface and in summary

- Differences to Seidel terms due to higher order at cemented surface for larger pupil radii

F/2 Achromat, f'=100

Ref: H. Zügge
Higher Order Aberrations

- Expansion approach for aberrations: cartesian product of invariants of rotational symmetry

\[ u = \frac{x^2 + y^2}{2}, \quad v = x \cdot x_p + y \cdot y_p, \quad w = \frac{x_p^2 + y_p^2}{2} \]

- Third order aberrations exponent sum 4

\[ W_{sph} = S \cdot (x_p^2 + y_p^2)^2 \]
\[ W_{coma} = C \cdot y \cdot y_p \cdot (x_p^2 + y_p^2) \]
\[ W_{ast} = A \cdot y^2 \cdot y_p^2 \]

- Fifth order aberrations exponent sum 6

\[ W_{sph} = S_{sph,zone} \cdot (x_p^2 + y_p^2)^3 \]
\[ W_{coma} = C_{linear} \cdot y \cdot y_p \cdot (x_p^2 + y_p^2)^2 \]
\[ W_{sph} = S_{sph,skew1} \cdot y^2 \cdot (x_p^2 + y_p^2)^2 \]
\[ W_{sph} = S_{sph,skew2} \cdot y^2 \cdot y_p^2 \cdot (x_p^2 + y_p^2) \]
\[ W_{coma} = C_{ell,coma1} \cdot y^3 \cdot y_p \cdot (x_p^2 + y_p^2)^3 \]
\[ W_{ptz} = C \cdot y^2 \cdot (x_p^2 + y_p^2) \]
\[ W_{dist} = D \cdot y^3 \cdot y_p \]
\[ W_{spP} = P \cdot y^4 \]

\[ W_{coma} = C_{ell,coma1} \cdot y^3 \cdot y_p^3 \]
\[ W_{ast} = A_{ast} \cdot y^4 \cdot y_p^2 \]
\[ W_{ptz} = C_{petz} \cdot y^4 \cdot (x_p^2 + y_p^2) \]
\[ W_{dist} = D_{dist} \cdot y^5 \cdot y_p \]
\[ W_{spP} = P_{sph,pup} \cdot y^6 \]
- Aberration expansion: perturbation theory

- Linear independent contributions only in lowest correction order:
  Surface contributions of Seidel additive

- Higher order aberrations (5th order,...): nonlinear superposition
  - 3rd order generates different ray heights and angles at next surfaces
  - induces aberration of 5th order
  - together with intrinsic surface contribution: complete error

- Separation of intrinsic and induced aberrations: refraction at every surface in the system
Surface No. j in the system:
- intermediate imaging with object, image, entrance and exit pupil
- separate calculations with ideal/real perturbed object point
- pupil distortion must be taken into account
Induced Aberrations

- Mathematical formulation:
  1. incoming aberrations form previous surface

\[ W_{\text{entr},j}(\vec{r}_p) = \sum_{i=1}^{j-1} [W^{(3)}_i(\vec{r}_p) + W^{(5)}_i(\vec{r}_p)] \]

  2. transfer into exit pupil

\[ W_{\text{exit},j}(\vec{r}_p) = \sum_{i=1}^{j-1} [W^{(3)}_i(\vec{r}_p + \delta r^{(3)}_{pj}) + W^{(5)}_i(\vec{r}_p)] + [W^{(3)}_j(\vec{r}_p) + W^{(5)}_j(\vec{r}_p)] \]

  3. complete/total aberration

\[ W_{\text{compl},j}(\vec{r}_p) = W_{\text{exit},j}(\vec{r}_p) - W_{\text{entr},j}(\vec{r}_p) \]

\[ = W^{(3)}_j(\vec{r}_p) + W^{(5)}_j(\vec{r}_p) + \sum_{i=1}^{j-1} W^{(3)}_i(\delta r^{(3)}_{pj}) \]

  4. subtraction total/intrinsic: induced aberrations

\[ W_{\text{induc},j}(\vec{r}_p) = \sum_{i=1}^{j-1} W^{(3)}_i(\delta r^{(3)}_{pj}) \]

- Interpretation:
  Induced aberration is generated by pupil distortion together with incoming perturbed 3rd order aberration

- Similar effects obtained for higher orders

- Usually induced aberrations are larger than intrinsic one
Induced Aberrations

- Example Gabor telescope
  - a lens pre-corrects a spherical mirror to obtain vanishing spherical aberration
  - due to the strong ray deviation at the plate, the ray heights at the mirror changes significantly
  - as a result, the mirror has induced chromatical aberration, also the intrinsic part is zero by definition

- Surface contributions and chromatic difference
Caustics

- Caustic phenomena in real world

Ref: J. Nye, Natural focusing
Caustics

- Early investigations on caustics: Leonardo da Vinci 1508
- Caustics at mirrors and lenses

Ref: J. Nye, Natural focusing
Caustics

- More general: caustic occurs at every wavefront with concave shape as locus of local curvature

- Physically:
  - crossing of rays indicates a caustic
  - interference with diffraction ripple and ringing is seen

Ref: J. Nye, Natural focusing
Ref: W. Singer
Caustics

- Caustic: envelope of rays
- Locus of local curvature
- Calculation:
  - Caustic: \( \vec{r}_c = (x_c, y_c, z_c) \)
  - Ray direction: \( \vec{s} = (s_x, s_y, s_z) \)
  - Rays: \( \vec{r}_c = \vec{r} + L \cdot \vec{s} \)
  - L distance PC

variation of point on wavefront:

solution condition for linear system:

\[
\begin{vmatrix}
1 + L \frac{\partial s_x}{\partial x} & L \frac{\partial s_x}{\partial y} & s_x \\
L \frac{\partial s_y}{\partial x} & 1 + L \frac{\partial s_y}{\partial y} & s_y \\
s_x & s_y & 1
\end{vmatrix} = 0
\]

\[
x_c = x + L \cdot s_x
\]
\[
y_c = y + L \cdot s_y
\]
\[
z_c = L \cdot s_z
\]

\[
\delta x + L \cdot \frac{\partial s_x}{\partial x} \cdot \delta x + L \cdot \frac{\partial s_x}{\partial y} \cdot \delta y + s_x \cdot \delta L = 0
\]
\[
\delta y + L \cdot \frac{\partial s_y}{\partial x} \cdot \delta x + L \cdot \frac{\partial s_y}{\partial y} \cdot \delta y + s_y \cdot \delta L = 0
\]
\[
L \cdot \frac{\partial s_z}{\partial x} \cdot \delta x + L \cdot \frac{\partial s_z}{\partial y} \cdot \delta y + s_z \cdot \delta L = 0
\]
Caustics

- Special case of one dimension $x$-$z$
- Example: spherical aberration for focussing through plane interface
- Ray direction

\[ s_x = \frac{\partial W}{\partial x} \]

- Variation

\[ \left( 1 + L \cdot \frac{\partial s_x}{\partial x} \right) \cdot \delta x + s_x \cdot \delta L = 0 \]

\[ L \cdot \frac{\partial s_z}{\partial x} \cdot \delta x + s_z \cdot \delta L = 0 \]

- Geometry and law of refraction

\[ s_x = \frac{\partial W}{\partial x} = -n \cdot \frac{x}{a} = -n \cdot \frac{x}{\sqrt{q^2 + x^2}} \]

\[ L = -\frac{s_z^2}{\partial s_x/\partial x} = \frac{a}{n} \left[ 1 - (n^2 - 1) \cdot \frac{x^2}{q^2} \right] \]

- Approximation of small $x$: caustic curve

\[ z_c = q \left[ \frac{1}{2n} \right] \cdot [(n^2 - 1) \cdot q]^{1/3} \cdot x_c^{2/3} \]


- Optical power flux
- Radiance $L$: power per area and solid angle
- General transfer: Jacobian matrix of differential area transform

\[
\Delta^2 P = \frac{L_j}{r_{j,j+1}^2} \cdot \cos \theta_j \cdot \cos \theta_{j+1} \cdot \Delta A_j \cdot \Delta A_{j+1}
\]

\[
\Delta A' = J \cdot \Delta A
\]

\[
J = \begin{vmatrix}
\frac{dx'}{dx} & \frac{dx'}{dy} \\
\frac{dx}{dx} & \frac{dy}{dy} \\
\frac{dy'}{dy} & \frac{dy'}{dx}
\end{vmatrix} = \frac{dx'}{dx} \cdot \frac{dy'}{dy} - \frac{dx'}{dy} \cdot \frac{dy'}{dx}
\]
Illumination in Optical Systems

- Consideration with the help of entrance and exit pupil:
  1. transfer from source to entrance pupil
  2. transfer between pupils
  3. transfer from exit pupil into image plane

- Important for illumination:
  1. aperture angle (vignetting)
  2. chief ray incidence angle
  3. chief ray intersection point (distortion)
  4. spreading of spot

\[
\frac{dA_{AP}}{dA_{EP}} = \left(\frac{n}{n'}\right)^2 \cdot \frac{dA}{dA'} \cdot \left(\frac{s_{EP}}{s_{AP}}\right)^2 \cdot \frac{\cos^4 w}{\cos^4 w'}
\]
Illumination

- Relation in the special case circular symmetry
  \[ E' = \pi L \cdot \sin^2 u \cdot k \cdot \frac{h}{h'} \cdot \frac{dh'}{dh} \cdot \cos^4 w \cdot f_{corr} \]

- Irradiance E:
- Numerical aperture \( \sin \theta \)
- Image location \( h, h' \)
- Distortion \( dh'/dh \)

Diagram:
- Object
- Entrance pupil
- Chief ray
- System
- Exit pupil
- Image
- \( dA, dA' \)
- \( dh, dh' \)
- \( d\psi, d\psi' \)
- \( h, h' \)
- \( s_p, s'_p \)
The complex field in the exit pupil determines the irradiance in the image.

The aberration and the pupil distortion influences the illumination brightness:
1. The pupil distortion changes the aperture cone size.
2. The image distortion changes the area element size and the location of the energy deposition.

The irradiance function can also be treated by a perturbation expansion analogous to the image aberrations.

The gradient of the wave aberrations determines the local Poynting vector and therefore influences the irradiance.

The sine condition is mainly considered for the fulfillment of a system without pupil aberration.

\[
E(\vec{H},\vec{r}_p) = \sqrt{I_o \cdot I(\vec{H},\vec{r}_p)} \cdot e^{-ik\cdot\left[n \cdot S(\vec{H},\vec{r}_p) + W(\vec{H},\vec{r}_p)\right]}
\]
Telecentric systems are used for applications with vanishing magnification changes during defocussing, they have large in diameter.

The Scheimpflug setup can sharply image tilted object planes.

The Scheimpflug imaging suffers from large distortion and anamorphic magnification.

Diffractive elements follow the local grating equation.

Arbitrary degrees of freedom of location and direction of ray bending.

Problems are unwanted diffraction orders and finite efficiency.

Color correction and vanishing field curvature are attractive.

Broadband chromatical correction is complicated.

Fresnel lenses are working refractive, but are similar to DOEs.

Fresnel lenses violate the sine condition and therefore suffer from coma.

Straylight is a problem in Fresnel lens systems.

Aldis theorem allows for a finite aberration surface contribution for one ray.

Higher order aberrations are important for larger angles of the chief or marginal ray.

Higher order perturbation theory is no longer linear independent.

The interaction of 3rd order and in particular pupil distortion generates induced aberrations.

Caustics are the locii of the local curvature centers.

At caustics, rays intersect and complicated envelopes are occurring.

Physically, interference produces a structured intensity at caustics.

Illumination systems can be considered with aberration theory too.

There are strong relations between aberrations theory and image irradiance.

In particular the sine condition fixes a distortion-free pupil transfer.