<table>
<thead>
<tr>
<th>No.</th>
<th>Date</th>
<th>Topic</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19.10.</td>
<td>Paraxial imaging</td>
<td>Paraxial optics, fundamental laws of geometrical imaging, compound systems</td>
</tr>
<tr>
<td>2</td>
<td>26.11.</td>
<td>Pupils, Fourier optics, Hamiltonian coordinates</td>
<td>Pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates</td>
</tr>
<tr>
<td>3</td>
<td>02.11.</td>
<td>Eikonal</td>
<td>Fermat principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media</td>
</tr>
<tr>
<td>4</td>
<td>09.11.</td>
<td>Aberration expansions</td>
<td>Single surface, general Taylor expansion, representations, various orders, stop shift formulas</td>
</tr>
<tr>
<td>5</td>
<td>16.11.</td>
<td>Representation of aberrations</td>
<td>Different types of representations, fields of application, limitations and pitfalls, measurement of aberrations</td>
</tr>
<tr>
<td>6</td>
<td>23.11.</td>
<td>Spherical aberration</td>
<td>Phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders</td>
</tr>
<tr>
<td>7</td>
<td>30.11.</td>
<td>Distortion and coma</td>
<td>Phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options</td>
</tr>
<tr>
<td>8</td>
<td>07.12.</td>
<td>Astigmatism and curvature</td>
<td>Phenomenology, Coddington equations, Petzval law, correction options</td>
</tr>
<tr>
<td>9</td>
<td>14.12.</td>
<td>Chromatical aberrations</td>
<td>Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum</td>
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<tr>
<td>10</td>
<td>21.12.</td>
<td>Sine condition, aplanatism and isoplanatism</td>
<td>Sine condition, isoplanatism, relation to coma and shift invariance, pupil aberrations, Herschel condition, relation to Fourier optics</td>
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<tr>
<td>11</td>
<td>04.01.</td>
<td>Wave aberrations</td>
<td>Definition, various expansion forms, propagation of wave aberrations</td>
</tr>
<tr>
<td>12</td>
<td>11.01.</td>
<td>Zernike polynomials</td>
<td>Special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, measurement</td>
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<tr>
<td>13</td>
<td>18.01.</td>
<td>Point spread function</td>
<td>Ideal psf, psf with aberrations, Strehl ratio</td>
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<tr>
<td>14</td>
<td>25.01.</td>
<td>Transfer function</td>
<td>Transfer function, resolution and contrast</td>
</tr>
<tr>
<td>15</td>
<td>01.02.</td>
<td>Additional topics</td>
<td>Vectorial aberrations, generalized surface contributions, Aldis theorem, intrinsic and induced aberrations, revertability</td>
</tr>
</tbody>
</table>
1. Definition of wave aberrations
2. Performance criteria
3. Primary aberrations
4. Order expansions
5. Non-circular pupil shapes
6. Statistical aberrations
7. Measurement of wave aberrations
Law of Malus-Dupin

- Law of Malus-Dupin:
  - equivalence of rays and wavefronts
  - both are orthonormal
  - identical information

- Condition:
  No caustic of rays

- Mathematical:
  Rotation of Eikonal vanish

\[ \text{rot}(\mathbf{n} \cdot \mathbf{s}) = 0 \]

- Optical system:
  Rays and spherical waves orthonormal
Relationship to Transverse Aberration

- Relation between wave and transverse aberration
- Approximation for small aberrations and small aperture angles $u$
- Ideal wavefront, reference sphere: $W_{\text{ideal}}$
- Real wavefront: $W_{\text{real}}$
- Finite difference

\[ \Delta W = W = W_{\text{real}} - W_{\text{ideal}} \]

\[ \varphi \approx \tan \varphi = \frac{\partial W}{\partial y_p} \]

\[ \Delta y' = -R \cdot \varphi \]

\[ \frac{\partial W}{\partial y_p} = - \frac{\Delta y'}{R - W} \approx - \frac{\Delta y'}{R} \]
Wave Aberration

- Exact relation between wave aberration and ray deviation
- General expression from geometry
  a describes the lateral aberration
- Substitution of angle by scalar product
- Exact relation is quadratic in R
- Approximation for large R

\[
\begin{align*}
W &= W_0 + n' \left( \frac{\vec{s}_r - \vec{s} \cdot (\vec{s} \cdot \vec{s}_r)}{1 + \vec{s} \cdot \vec{s}_r} \right) \cdot \vec{a} + n' \cdot \frac{a^2 - (\vec{s} \cdot \vec{a})^2}{R \cdot (1 + \cos \theta)} \\
W &= W_\infty + n' \cdot \frac{a^2 - (\vec{s} \cdot \vec{a})^2}{2R} \cdot \left[ 1 + \frac{a^2 - (\vec{s} \cdot \vec{a})^2}{4R^2} \right]
\end{align*}
\]

\[
\Delta W = -\frac{x_p}{R} \cdot \Delta x' - \frac{y_p}{R} \cdot \Delta y' - \frac{x_p^2 + y_p^2}{2 \cdot R} \cdot \Delta z'
\]

with

\[
\vec{a} = \begin{pmatrix} 
\Delta x' \\
\Delta y' \\
\Delta z'
\end{pmatrix}
\]
Relationships

- Concrete calculation of wave aberration: addition of discrete optical path lengths (OPL)

- Reference on chief ray and reference sphere (optical path difference)

- Relation to transverse aberrations

- Conversion between longitudinal transverse and wave aberrations

- Scaling of the phase / wave aberration:
  1. Phase angle in radiant
  2. Light path (OPL) in mm
  3. Light path scaled in $\lambda$

$$l_{OPL} = \int_{OEP}^{AP} n \cdot d\vec{r}$$

$$\Delta_{OPD}(x, y) = l_{OPL}(x, y) - l_{OPL}(0,0)$$

$$\frac{\partial W}{\partial y_p} = -\frac{\Delta y'}{R-W} \approx -\frac{\Delta y'}{R}$$

$$\Delta s' = \frac{R}{y_p} \cdot \Delta y' = \frac{\Delta y'}{\sin u'} = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p}$$

$$E(x) = A(x) \cdot e^{i \cdot \phi(x)}$$

$$E(x) = A(x) \cdot e^{i \cdot k \cdot \Delta_{OPD}(x)}$$

$$E(x) = A(x) \cdot e^{2\pi i \cdot W(x)}$$
Definition of optical path length in an optical system:
Reference sphere around the ideal object point through the center of the pupil
Chief ray serves as reference
Difference of OPL: optical path difference OPD
Practical calculation: discrete sampling of the pupil area, real wave surface represented as matrix
Pupil Sampling

- All rays start in one point in the object plane
- The entrance pupil is sampled equidistant
- In the exit pupil, the transferred grid may be distorted
- In the image plane a spreaded spot diagram is generated
Wave Aberration

- Definition of the peak valley value $W_{PV}$
- Reference sphere corresponds to perfect imaging
- Rms-value is more relevant for performance evaluation
Wave Aberration Criteria

- Mean quadratic wave deviation (\(W_{\text{Rms}}\), root mean square)

\[
W_{\text{rms}} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \sqrt{\frac{1}{A_{\text{ExP}}} \iint [W(x_p, y_p) - W_{\text{mean}}(x_p, y_p)]^2 \, dx_p \, dy_p}
\]

with pupil area

\[
A_{\text{ExP}} = \iint \, dx \, dy
\]

- Peak valley value \(W_{pv}\): largest difference

\[
W_{pv} = \max\{W_{\text{max}}(x_p, y_p) - W_{\text{min}}(x_p, y_p)\}
\]

- General case with apodization:
weighting of local phase errors with intensity, relevance for psf formation

\[
W_{\text{rms}} = \sqrt{\frac{1}{A_{\text{ExP}}^{(w)}} \iint I_{\text{ExP}}(x_p, y_p) \cdot [W(x_p, y_p) - W_{\text{mean}}^{(w)}(x_p, y_p)]^2 \, dx_p \, dy_p}
\]
Wave Aberrations – Sign and Reference

- Wave aberration: relative to reference sphere
- Choice of offset value: vanishing mean
- Sign of W:
  - W > 0: stronger convergence, intersection: s < 0
  - W < 0: stronger divergence, intersection: s < 0

\[
\langle W(x, y) \rangle = \frac{1}{F_{Exp}} \iint W(x, y) \, dx \, dy = 0
\]
Tilt of Wavefront

- Change of reference sphere: tilt by angle $\theta$
  
  linear in $y_p$

  $$\Delta W_{\text{tilt}} = n \cdot y_p \cdot \theta$$

- Wave aberration due to transverse aberration $\Delta y'$

  $$\Delta W_{\text{tilt}} = -\frac{y_p}{R_{\text{Ref}}} \cdot \Delta y'$$

- Is the usual description of distortion
Defocussing of Wavefront

Paraxial defocussing by \( \Delta z \):
Change of wavefront

\[
\Delta W_{Def} = -\frac{n \cdot r_p^2}{2R_{ref}^2} \Delta z' = -\frac{1}{2} n \cdot \Delta z' \cdot \sin^2 u
\]
Primary Aberrations

- Representation of primary aberrations
  - Seidel terms
- Surface in pupil plane
- Special case of chromatical aberrations

Primary monochromatic wave aberrations

<table>
<thead>
<tr>
<th></th>
<th>Spherical aberration</th>
<th>Coma</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W = c_1 \cdot (x_p^2 + y_p^2)$</td>
<td>$W = c_2 \cdot y_p \cdot (x_p^2 + y_p^2)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Astigmatism</th>
<th>Field curvature (sagittal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W = c_3 \cdot y^2 y_p^2$</td>
<td>$W = c_4 \cdot y^2 (x_p^2 + y_p^2)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W = c_5 \cdot y^3 y_p$</td>
<td></td>
</tr>
</tbody>
</table>

Ref: H. Zügge
Primary Aberrations

- Relation: wave / geometrical aberration

<table>
<thead>
<tr>
<th>Type</th>
<th>Wave aberration</th>
<th>Geometrical spot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical aberration</td>
<td>$W = c_1 \cdot r^4$</td>
<td>$\Delta x' \propto c_1 \cdot r^3 \sin \varphi$ $\Delta y' \propto c_1 \cdot r^3 \cos \varphi$</td>
</tr>
<tr>
<td>Coma</td>
<td>$W = c_2 \cdot yr^2 \cos \varphi$</td>
<td>$\Delta y' \propto c_2 \cdot yr^2 \cdot (2 + \cos 2 \varphi)$ $\Delta x' \propto c_2 \cdot yr^2 \sin 2 \varphi$</td>
</tr>
<tr>
<td>Astigmatism</td>
<td>$W = c_3 \cdot y^2r^2 \cos^2 \varphi$</td>
<td>$\Delta x' = 0$ $\Delta y' \propto c_3 \cdot y^2r \cos \varphi$</td>
</tr>
<tr>
<td>Field curvature</td>
<td>$W = c_4 \cdot y^2r^2$</td>
<td>$\Delta x' \propto c_4 \cdot y^2r \sin \varphi$ $\Delta y' \propto c_4 \cdot y^2r \cos \varphi$</td>
</tr>
<tr>
<td>Distortion</td>
<td>$W = c_5 \cdot y^3r \cos \varphi$</td>
<td>$\Delta x' = 0$ $\Delta y' \propto c_5 \cdot y^3$</td>
</tr>
</tbody>
</table>

Ref: H. Zügge
Primary Aberrations

- Relation: wave / geometrical aberration

<table>
<thead>
<tr>
<th>Type</th>
<th>Wave aberration</th>
<th>Geometrical spot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial color</td>
<td>$\delta W = \tilde{b}_1 \cdot r^2$</td>
<td>$\Delta x' \propto \tilde{b}_1 \cdot r \sin \varphi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta y' \propto \tilde{b}_1 \cdot r \cos \varphi$</td>
</tr>
<tr>
<td>Symmetry to Periodicity</td>
<td></td>
<td>one plane 1 period</td>
</tr>
<tr>
<td>Lateral color</td>
<td>$\delta W = \tilde{b}_2 \cdot y r \cos \varphi$</td>
<td>$\Delta x' = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta y' \propto \tilde{b}_2 \cdot y$</td>
</tr>
<tr>
<td>Symmetry to Periodicity</td>
<td></td>
<td>one straight line constant</td>
</tr>
</tbody>
</table>

Ref: H. Zügge
Special Cases of Wave Aberrations

- Wave aberrations are usually given as reduced aberrations:
  - wave front for only 1 field point
  - field dependence represented by discrete cases

- Special case of aberrations:
  1. axial color and field curvature: represented as defocussing term, Zernike $c_4$
  2. distortion and lateral color: represented as tilt term, Zernike $c_2, c_3$
3. afocal system
   - exit pupil in infinity
   - plane wave as reference

4. telecentric system
   chief ray parallel to axis
Expulsion of the Wave Aberration

- Table as function of field and aperture
- Selection rules:
  checkerboard filling of the matrix

<table>
<thead>
<tr>
<th>Aperture $r$</th>
<th>Field $y$</th>
<th>Spherical $y^0$</th>
<th>Coma $y^1$</th>
<th>Astigmatism $y^2$</th>
<th>$y^3$</th>
<th>$y^4$</th>
<th>$y^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^1$</td>
<td>Distortion</td>
<td>$y r \cos \theta$ Tilt</td>
<td>$y^3 r \cos \theta$ Distortion primary</td>
<td>$y^5 r \cos \theta$ Distortion secondary</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^2$</td>
<td>Defocus</td>
<td>$y^2 r^2 \cos^2 \theta$ Astig./Curvat.</td>
<td>$y^4 r^2 \cos^2 \theta$</td>
<td>$y^4 r^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^3$</td>
<td></td>
<td>$y r^3 \cos \theta$ Coma primary</td>
<td>$y^3 r^3 \cos^3 \theta$</td>
<td>$y^3 r^3 \cos^3 \theta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^4$</td>
<td>Spherical primary</td>
<td>$y^2 r^4 \cos^2 \theta$</td>
<td>$y^2 r^4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^5$</td>
<td></td>
<td>$y r^5 \cos \theta$ Coma secondary</td>
<td>$y^2 r^4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^6$</td>
<td>Spherical secondary</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Image location

Primary aberrations / Seidel

Secondary aberrations
Polynomial Expansion of Wave Aberrations

- Taylor expansion of the wavefront:
  \[ W(y', r_p, \theta) = \sum_{k,l,m} W_{klm} y'^k r_p^l \cos^m \theta \]
  - \( y' \) Image height
  - \( r_p \) Pupil height
  - \( \theta \) Pupil azimuth angle

- Symmetry invariance:
  1. Image height
  2. Pupil height
  3. Scalar product between image and pupil vector

- Number of terms
  sum of indices in the exponent \( i_{\text{sum}} \)

<table>
<thead>
<tr>
<th>( i_{\text{sum}} )</th>
<th>( N_i ) number of terms</th>
<th>Type of aberration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>image location</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>primary aberrations, 4th order</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>secondary aberrations, 6th order</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>8th order</td>
</tr>
</tbody>
</table>
The exponents of the Taylor expansion on the aperture depends on the kind of representation of the aberrations.

- The exponent grows by 1 in the sequence longitudinal-transversal-wave aberrations.
- The Seidel term '3rd order' is valid only for transverse aberrations.
- Dependence on aperture and field size for the primary aberrations:

<table>
<thead>
<tr>
<th>type of aberration</th>
<th>wave aberration</th>
<th>transverse aberration</th>
<th>longitudinal aberration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>u w</td>
<td>u w</td>
<td>u w</td>
</tr>
<tr>
<td>spherical</td>
<td>4 0</td>
<td>3 0</td>
<td>2 0</td>
</tr>
<tr>
<td>coma</td>
<td>3 1</td>
<td>2 1</td>
<td></td>
</tr>
<tr>
<td>astigmatism</td>
<td>2 2</td>
<td>1 2</td>
<td>0 2</td>
</tr>
<tr>
<td>Petzval curvature</td>
<td>2 2</td>
<td>1 2</td>
<td>0 2</td>
</tr>
<tr>
<td>distortion</td>
<td>1 3</td>
<td>0 3</td>
<td></td>
</tr>
<tr>
<td>axial chromatical</td>
<td>2 0</td>
<td>1 0</td>
<td>0 0</td>
</tr>
<tr>
<td>chromatical magnif.</td>
<td>1 1</td>
<td>0 1</td>
<td></td>
</tr>
</tbody>
</table>
Taylor Expansion of the Primary Aberrations

- Expansion of the monochromatic aberrations
- First real aberration: primary aberrations, 4\textsuperscript{th} order as wave deviation

\[ W(y', r_p, y_p) = A_s r_p^4 + A_c y' r_p^2 y_p + A_a y'^2 y_p^2 + A_p y'^2 r_p^2 + A_d y'^3 y_p \]

- Coefficients of the primary aberrations:
  \( A_s \) : Spherical Aberration
  \( A_c \) : Coma
  \( A_a \) : Astigmatism
  \( A_p \) : Petzval curvature
  \( A_d \) : Distortion

- Alternatively: expansion in polar coordinates:
  Zernike basis expansion, usually only for one field point, orthogonalized
Rayleigh Criterion

- The Rayleigh criterion \( |W_{PV}| \leq \frac{\lambda}{4} \)
gives individual maximum aberrations coefficients, depends on the form of the wave

- Examples:

<table>
<thead>
<tr>
<th>aberration type</th>
<th>coefficient</th>
<th>aberration type</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus</td>
<td>Seidel</td>
<td>defocus</td>
<td>Zernike</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Seidel</td>
<td>spherical aberration</td>
<td>Zernike</td>
</tr>
<tr>
<td>astigmatism</td>
<td>Seidel</td>
<td>astigmatism</td>
<td>Zernike</td>
</tr>
<tr>
<td>coma</td>
<td>Seidel</td>
<td>coma</td>
<td>Zernike</td>
</tr>
</tbody>
</table>

- a) optimal constructive interference
- b) reduced constructive interference due to phase aberrations
- c) reduced effect of phase error by apodization and lower energetic weighting
- d) start of destructive interference for 90° or \( \lambda/4 \) phase aberration begin of negative z-component
Criteria of Rayleigh and Marechal

- **Rayleigh criterion:**
  1. maximum of wave aberration: $W_{pv} < \lambda/4$
  2. beginning of destructive interference of partial waves
  3. limit for being diffraction limited (definition)
  4. as a PV-criterion rather conservative: maximum value only in 1 point of the pupil
  5. different limiting values for aberration shapes and definitions (Seidel, Zernike,...)

- **Marechal criterion:**
  1. Rayleigh criterion corresponds to $W_{rms} < \lambda/14$ in case of defocus

\[
W_{rms}^{Rayleigh} \leq \frac{\lambda}{\sqrt{192}} \approx \frac{\lambda}{13.856} \approx \frac{\lambda}{14}
\]

  2. generalization of $W_{rms} < \lambda/14$ for all shapes of wave fronts
  3. corresponds to Strehl ratio $D_s > 0.80$ (in case of defocus)
  4. more useful as PV-criterion of Rayleigh
PV and $W_{rms}$-Values

- PV and $W_{rms}$ values for different definitions and shapes of the aberrated wavefront

- Due to mixing of lower orders in the definition of the Zernikes, the $W_{rms}$ usually is smaller in comparison to the corresponding Seidel definition

<table>
<thead>
<tr>
<th>aberration type</th>
<th>definition</th>
<th>mean $W_{mean}$</th>
<th>peak-valley $W_{pv}$</th>
<th>root mean square $W_{rms}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus Seidel</td>
<td>$a_{20} \cdot r_p^2$</td>
<td>$\frac{a_{20}}{2}$</td>
<td>$a_{20}$</td>
<td>$\frac{a_{20}}{2\sqrt{3}} = 0.289 \cdot a_{20}$</td>
</tr>
<tr>
<td>defocus Zernike</td>
<td>$c_{20} \cdot \left(2r_p^2 - 1\right)$</td>
<td>0</td>
<td>$2c_{20}$</td>
<td>$\frac{c_{20}}{\sqrt{3}} = 0.577 \cdot c_{20}$</td>
</tr>
<tr>
<td>spherical aberration Seidel</td>
<td>$a_{40} \cdot r_p^4$</td>
<td>$\frac{a_{40}}{3}$</td>
<td>$a_{40}$</td>
<td>$\frac{2a_{40}}{3\sqrt{5}} = 0.298 \cdot a_{40}$</td>
</tr>
<tr>
<td>spherical aberration with defocus</td>
<td>$b_{40} \cdot \left(r_p^4 - r_p^2\right)$</td>
<td>$-\frac{b_{40}}{6}$</td>
<td>$\frac{b_{40}}{4}$</td>
<td>$\frac{b_{40}}{6\sqrt{5}} = 0.075 \cdot b_{40}$</td>
</tr>
<tr>
<td>spherical aberration Zernike</td>
<td>$c_{40} \cdot \left(6r_p^4 - 6r_p^2 + 1\right)$</td>
<td>0</td>
<td>$\frac{3c_{40}}{\sqrt{5}}$</td>
<td>$\frac{c_{40}}{\sqrt{5}} = 0.447 \cdot c_{40}$</td>
</tr>
<tr>
<td>astigmatism Seidel</td>
<td>$a_{22} \cdot r_p^2 \cos^2 \theta$</td>
<td>$\frac{a_{22}}{4}$</td>
<td>$a_{22}$</td>
<td>$\frac{a_{22}}{4} = 0.25 \cdot a_{22}$</td>
</tr>
<tr>
<td>astigmatism with defocus</td>
<td>$b_{22} \left(r_p^2 \cos^2 \theta - \frac{1}{2} r_p^2\right)$</td>
<td>0</td>
<td>$b_{22}$</td>
<td>$\frac{b_{22}}{2\sqrt{6}} = 0.204 \cdot b_{22}$</td>
</tr>
<tr>
<td>astigmatism Zernike</td>
<td>$c_{22} \left(2r_p^2 \cos^2 \theta - r_p^2\right)$</td>
<td>0</td>
<td>$2c_{22}$</td>
<td>$\frac{c_{22}}{\sqrt{6}} = 0.408 \cdot c_{22}$</td>
</tr>
<tr>
<td>coma Seidel</td>
<td>$a_{31} \cdot r_p^3 \cos \theta$</td>
<td>0</td>
<td>$2a_{31}$</td>
<td>$\frac{a_{31}}{2\sqrt{2}} = 0.353 \cdot a_{31}$</td>
</tr>
<tr>
<td>coma with tilt</td>
<td>$b_{31} \left(r_p^3 - \frac{2}{3} r_p\right) \cos \theta$</td>
<td>$0$</td>
<td>$\frac{2b_{31}}{3}$</td>
<td>$\frac{b_{31}}{6\sqrt{2}} = 0.118 \cdot b_{31}$</td>
</tr>
<tr>
<td>coma Zernike</td>
<td>$c_{31} \left(3r_p^3 - 2r_p\right) \cos \theta$</td>
<td>0</td>
<td>$2c_{31}$</td>
<td>$\frac{c_{31}}{2\sqrt{2}} = 0.353 \cdot c_{31}$</td>
</tr>
</tbody>
</table>
Typical Variation of Wave Aberrations

- Microscopic objective lens

- Changes of rms value of wave aberration with
  1. wavelength
  2. field position

- Common practice:
  1. diffraction limited on axis for main part of the spectrum
  2. Requirements relaxed in the outer field region
  3. Requirement relaxed at the blue edge of the spectrum

![Graph showing changes in wave aberrations with wavelength and field position.]
Zernike Polynomials

- Expansion of wave aberration surface

\[ W(r, \varphi) = \sum_{n} \sum_{m=-n}^{n} c_{nm} Z_n^m(r, \varphi) \]

- Zernike polynomials orders by indices:
  - \( n \) : radial
  - \( m \) : azimuthal, \( \sin/\cos \)

- Orthonormal function on unit circle

\[ Z_n^m(r, \varphi) = R_n^m(r) \cdot \begin{cases} 
\sin(m\varphi) & \text{for } m < 0 \\
\cos(m\varphi) & \text{for } m > 0 \\
1 & \text{for } m = 0
\end{cases} \]

- Direct relation to primary aberration types
- Direct measurement by interferometry
- Orthogonality perturbed:
  1. apodization
  2. discretization
  3. real non-circular boundary
Tatian Polynomials for Ring Pupils

- Orthogonalization of Zernike Polynomials for ring shaped pupil area

- Basis function depends on obsuration parameter $e$: no easy comparisons possible
Systems with rectangular pupil:
Use of Legendre polynomials $P_n(x)$

1. Factorized representation
   Problem: zero-crossing lines
2. Definition of 2D area-orthogonal Legendre functions

General shape of the pupil area:
Gram-Schmidt-orthogonalization drawback:
1. Individual function for every pupil shape
2. no intuitive interpretation
3. no comparability between different systems possible

\[
\int_{-1}^{+1} P_n(x) \cdot P_m(x) \, dx = \begin{cases} 
0 & \text{if } n \neq m \\
2 & \text{if } n = m \\
2n + 1 & \text{if } n = m
\end{cases}
\]

\[
W(x, y) = \sum_{n} \sum_{m} A_{nm} P_n(x) \cdot P_m(y)
\]
2-Dimensional Legendre Polynomials

- Orthogonalization of Zernike polynomials on a unit square

\[
\int_{-1}^{1} \int_{-1}^{1} Q_k Q_j \, dx \, dy = \delta_{kj} \quad \text{with} \quad Q_m = \sum_{j=0}^{j_{\text{max}}} \sum_{l=0}^{l_{\text{max}}} d_{m,jl} \cdot x^j y^l \quad \text{and}\quad Q_m = \sum_{j=1}^{m} c_{mj} \cdot Z_j
\]

- Gram-Schmidt-Orthogonalization procedure

\[
Q_m = \sum_{k=1}^{m-1} a_k \cdot Q_k + a_m \cdot Z_m
\]

\[
c_{mk} = \sum_{j=k}^{m-1} a_j \cdot c_{jk} \quad k = 1, 2, 3, \ldots m - 1
\]

\[
a_m = c_{mm} = \frac{1}{\sqrt{P_{mm} + \sum_{k=1}^{m-1} T_{km}^2 - 2 \cdot \sum_{k=1}^{m-1} T_{km} \sum_{j=1}^{k} c_{kj} \cdot P_{mj}}}
\]
- 2D-Legendre polynomials for rectangular areas
- Application: Spectrometer slit aperture
- First few polynomials: quite similar to Zernikes
Complex field with statistical phase

Correlation of the phase: structural function

Coherence function

For gaussian statistics

Auto covariance function

PSD, power spectral density

\[ E(\vec{r}) = E_o(\vec{r})e^{i\Phi(\vec{r})} \]

\[ e^{-\frac{1}{2}D_{\Phi}(\vec{r}_{12})} = \langle e^{i\Phi(\vec{r}_1) - i\Phi(\vec{r}_2)} \rangle \]

\[ \Gamma_{12}(z) = E_o(\vec{r}_1)E_o(\vec{r}_2)e^{-\frac{1}{2}D_{\Phi}(\vec{r}_{12})} \]

\[ D_{\Phi}(r) = 2\sigma^2 \left( 1 - e^{-\frac{r^2}{a_c^2}} \right) \]

\[ C(\Delta x, \Delta y) = \langle \Phi(x, y)\Phi(x + \Delta x, y + \Delta y) \rangle = \sigma^2 e^{-\frac{x^2 + y^2}{a_c^2}} \]

\[ S(v_x, v_y) = \hat{F}[C(x, y)] = \pi\sigma^2 a_c^2 e^{-\pi^2 a_c^2 (v_x^2 + v_y^2)} \]
Description:
1. in the spatial domain: topology of the rough surface
2. in the spatial frequency domain
- Power spectral density of the perturbation
- Three typical frequency ranges, scaled by diameter D

![Graph showing spatial frequency of surface perturbations with different frequency ranges and their corresponding power spectral density.](image)
Atmospheric Turbulence

- Atmospheric turbulence: statistical phase screen

- Scale below 1 cm: Tatarski regime, viscosity PSD
  \[ \Phi = b_{Ta} \cdot e^{-\left( \frac{k}{k_i} \right)^2} \]

- Scale from 1 cm to 5 m: Kolmogorov regime PSD
  \[ \Phi = b_{Ta} \cdot k^{-\frac{11}{3}} \]

- Greater length scales: Karman regime PSD
  \[ \Phi = b_{Ka} \cdot \left( k^2 + \frac{4\pi^2}{L_o^2} \right)^{-\frac{11}{6}} \]
Measurement of Wave Aberrations

- Wave aberrations are measurable directly
- Good connection between simulation/optical design and realization/metrology
- Direct phase measuring techniques:
  1. Interferometry
  2. Hartmann-Shack
  3. Hartmann sensor
  4. Special: Moire, Holography, phase-space analyzer
- Indirect measurement by inversion of the wave equation:
  1. Phase retrieval of PSF z-stack
  2. Retrieval of edge or line images
- Indirect measurement by analyzing the imaging conditions:
  from general image degradation
- Accuracy:
  1. $\lambda/1000$ possible, $\lambda/100$ standard for rms-value
  2. Rms vs. individual Zernike coefficients
Testing with Twyman-Green Interferometer

- Short common path, sensible setup
- Two different operation modes for reflection or transmission
- Always factor of 2 between detected wave and component under test

1. mode:
lens tested in transmission
auxiliary mirror for auto-collimation

2. mode:
surface tested in reflection
auxiliary lens to generate convergent beam
Interferograms of Primary Aberrations

Spherical aberration $1 \lambda$

Astigmatism $1 \lambda$

Coma $1 \lambda$

Defocussing in $\lambda$

-1  -0.5  0  +0.5  +1
Problems in real world measurement:

- Edge effects
  Definition of boundary

- Perturbation by coherent stray light

- Local surface error are not well described by Zernike expansion

- Convolution with motion blur

Ref: B. Dörband
Critical definition of the interferogram boundary and the Zernike normalization radius in reality
- Wave front determines local direction of propagation
- Propagation over distance $z$: change of transverse intensity distribution
- Intensity propagation contains phase information
Transport of intensity equation couples phase and intensity

\[ k \cdot \frac{\partial I(x, y, z)}{\partial z} = -\nabla [I(x, y, z) \cdot \nabla W(x, y)] \]

Solution with z-variation of the intensity delivers start phase at \( z = 0 \)

Determine phase from intensity distribution.
- Inverse propagation problem: ill posed
- Boundary condition: measured z-stack \( I(x, y, z) \)

Algorithm for numerical solution
- IFTA / Gerchberg Saxton (error reduction)
- Acceleration (conjugate gradients, Fienup,...)
- Modal non least square-methods
- Extended Zernike method

Applications:
- Calculation of diffractive components for given illumination distribution
- Wave front reconstruction
- Phase microscopy
Phase Retrieval

- Principle of phase retrieval for metrology of optical systems
- Measurement of intensity caustic z-stack
- Reconstruction of the phase in the exit pupil
Gerchberg-Saxton-Algorithm

- Iterative reconstruction of the pupil phase with back-and-forth calculation between image and pupil: IFTA / Gerchberg-Saxton
- Substitution of known intensity
- Problems with convergence: Twin-image degeneration
- Modified algorithms:
  1. Fienup-acceleration
  2. Non-least-square
  3. Use of pupil intensity
Phase Space Interpretation

- Known measurement of intensity in defocussed planes:
  - Several rotated planes in phase space
  - Information in and near the spatial domain
- Calculation of distribution in the Fourier plane
- Wave equation is valid
- Principle: Tomography
Example Phase Retrieval

- Evaluation of real data psf-stack
Phase Retrieval with Apodization

- Retrieval without / with Apodization
- Correlation over z
Object Space Defocussing

a) defocussing in image space

b) defocussing in object space
Measurements

- Comparison phase retrieval vs Hartmann test
- Case of coma

Ref: B. Möller