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0. Introduction

- ‘optique’ (Greek) ➔ lore of light ➔ ‘what is light’?
- Is light a wave or a particle (photon)?

- Light is the origin and requirement for life ➔ photosynthesis
- 90% of information we get is visual

A) What is light?
- electromagnetic wave \( (c = 3 \times 10^8 \text{ m/s}) \)
- amplitude and phase ➔ complex description
- polarization, coherence

<table>
<thead>
<tr>
<th>Region</th>
<th>Wavelength [nm]</th>
<th>Wavelength [m] (nm=10^{-9}m)</th>
<th>Frequency [Hz] (THz=10^{12}Hz)</th>
<th>Energy [eV]</th>
</tr>
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<tbody>
<tr>
<td>Radio</td>
<td>&gt; 10^8</td>
<td>&gt; 10^{-1}</td>
<td>&lt; 3 \times 10^{12}</td>
<td>&lt; 10^{-5}</td>
</tr>
<tr>
<td>Microwave</td>
<td>10^8 - 10^9</td>
<td>10^{-1} - 10^{-4}</td>
<td>3 \times 10^{10} - 3 \times 10^{12}</td>
<td>10^5 - 0.01</td>
</tr>
<tr>
<td>Infrared</td>
<td>10^5 - 700</td>
<td>10^{-6} - 7 \times 10^{-4}</td>
<td>3 \times 10^{13} - 4.3 \times 10^{14}</td>
<td>0.01 - 2</td>
</tr>
<tr>
<td>Visible</td>
<td>700 - 400</td>
<td>7 \times 10^{-7} - 4 \times 10^{-7}</td>
<td>4.3 \times 10^{14} - 7.5 \times 10^{14}</td>
<td>2 - 3</td>
</tr>
<tr>
<td>Ultraviolet</td>
<td>400 - 1</td>
<td>4 \times 10^{-7} - 10^{-9}</td>
<td>7.5 \times 10^{14} - 3 \times 10^{17}</td>
<td>3 - 10^{-1}</td>
</tr>
<tr>
<td>X-Rays</td>
<td>1 - 0.01</td>
<td>10^{-9} - 10^{-11}</td>
<td>3 \times 10^{17} - 3 \times 10^{19}</td>
<td>10^3 - 10^5</td>
</tr>
<tr>
<td>Gamma Rays</td>
<td>&lt; 0.01</td>
<td>&lt; 10^{-11}</td>
<td>&gt; 3 \times 10^{18}</td>
<td>&gt; 10^5</td>
</tr>
</tbody>
</table>
B) Origin of light
- atomic system \(\rightarrow\) determines properties of light (e.g. statistics, frequency, line width)
- optical system \(\rightarrow\) other properties of light (e.g. intensity, duration, …)
- invention of laser in 1958 \(\rightarrow\) very important development

- laser \(\rightarrow\) artificial light source with new and unmatched properties (e.g. coherent, directed, focused, monochromatic)
- applications of laser: fiber-communication, DVD, surgery, microscopy, material processing, …
D) Light can modify matter
- light induces physical, chemical and biological processes
- used for lithography, material processing, or modification of biological objects (bio-photonics)

E) Optical telecommunication
- transmitting data (Terabit/s in one fiber) over transatlantic distances

1000 m telecommunication fiber is installed every second.
F) **Optics in medicine and life sciences**

![Optical images](image)

Figure 1. Rotation of an intracellular object inside *Elodea densa* plant cell using the rotating line tweezers. The rod shaped structure was trapped using 25 mW power and rotated at a speed of 4 Hz. The direction of rotation is shown by arrow (a). Clockwise rotation by angles of 45° (b), 145° (c), and 235° (d). All the images were recorded with the same magnification.

G) **Light sensors and light sources**

- new light sources to reduce energy consumption
- new projection techniques

[Deutscher Zukunftspreis 2008 - IOF Jena + OSRAM]
H) Micro- and nano-optics
- ultra small camera

Insect inspired camera system develop at Fraunhofer Institute IOF Jena

I) Relativistic optics

Figure 3. Two relativistic lasers: (a) Holocos circa 1980 at LLE at LBNL was the first relativistic laser with $\omega_0 \approx 1$ at a millihertz repetition rate. (Courtesy of LBNL) (b) The $\lambda^3$ laser at the University of Michigan has an $\omega_0 \approx 1$ at a kilohertz repetition rate.

Figure 4. Relativistic rectification in plasma: (a) a high-intensity pulse before it enters the plasma and (b) the $E \times B$ that push the first plasma electrons. The electrons drag the heavy ions behind them like a horse pulling a cart. The electrostatic field that is generated is almost as large as the transverse laser field.
J) Schematic of optics

- geometrical optics
  - $\lambda \ll \text{size of objects} \rightarrow \text{daily experience}$
  - optical instruments, optical imaging
    - intensity, direction, coherence, phase, polarization, photons
    - G: Intensität, Richtung, Kohärenz, Phase, Polarisations, Photonen

- wave optics
  - $\lambda \approx \text{size of objects} \rightarrow \text{interference, diffraction, dispersion, coherence}$
  - laser, holography, resolution, pulse propagation
    - intensity, direction, coherence, phase, polarization, photons

- electromagnetic optics
  - reflection, transmission, guided waves, resonators
  - laser, integrated optics, photonic crystals, Bragg mirrors ...
  - intensity, direction, coherence, phase, polarization, photons

- quantum optics
  - small number of photons, fluctuations, light-matter interaction
    - intensity, direction, coherence, phase, polarization, photons

- in this lecture
  - electromagnetic optics and wave optics

- no quantum optics $\rightarrow$ advanced lecture

K) Literature

- Fundamental
     in German: "Grundlagen der Photonik" Wiley (2008)
  2. Hecht, 'Optic', Addison-Wesley (2001)
     in German: "Optik", Oldenbourg (2005)
  7. Sommerfeld, 'Optik'

- Advanced
  1. W. Silvast, 'Laser Fundamentals',
  4. Karthe, Müller, 'Integrierte Optik', Teubner
  5. Diels, Rudolph, 'Ultrashort Laser Pulse Phenomena', Academic
  7. Snyder, Love, 'Optical Waveguide Theory', Chapman&Hall
1. Ray optics - geometrical optics

1.1 Introduction
- Ray optics or geometrical optics is the simplest theory for doing optics.
- In this theory, propagation of light in various optical media can be described by simple geometrical rules.
- Ray optics is based on a very rough approximation ($\lambda \rightarrow 0$, no wave phenomena), but we can explain almost all daily life experiences involving light (shadows, mirrors, etc.).
- In particular, we can describe optical imaging with ray optics approach.
- In isotropic media, the direction of rays corresponds to the direction of energy flow.

What is covered in this chapter?
- It gives fundamental postulates of the theory.
- It derives simple rules for propagation of light (rays).
- It introduces simple optical components.
- It introduces light propagation in inhomogeneous media (graded-index (GRIN) optics).
- It introduces paraxial matrix optics.

1.2 Postulates
A) Light propagates as rays. Those rays are emitted by light-sources and are observable by optical detectors.
B) The optical medium is characterized by a function $n(r)$, the so-called refractive index ($n(r) \geq 1 -$ meta-materials $n(r) < 0$)
\[ n = \frac{c}{c_n} \quad c_n \text{ - speed of light in the medium} \]
C) optical path length $\sim$ delay
   i) homogeneous media
   \[ \int n(r) ds \]
   ii) inhomogeneous media
   \[ \int n(r) ds \]
D) Fermat’s principle
\[ \delta \int n(r) ds = 0 \]
Rays of light choose the optical path with the shortest delay.

1.3 Simple rules for propagation of light
A) Homogeneous media
   - $n = \text{const.} \rightarrow$ minimum delay = minimum distance
   - Rays of light propagate on straight lines.
B) Reflection by a mirror (metal, dielectric coating)
   - The reflected ray lies in the plane of incidence.
   - The angle of reflection equals the angle of incidence.
C) Reflection and refraction by an interface
   - Incident ray $\rightarrow$ reflected ray plus refracted ray
   - The reflected ray obeys b).
   - The refracted ray lies in the plane of incidence.
   - The angle of refraction $\theta_2$ depends on the angle of incidence $\theta_1$ and is given by Snell’s law:
     \[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]
   - no information about amplitude ratio.

1.4 Simple optical components
A) Mirror
   i) Planar mirror
   - Rays originating from $P_1$ are reflected and seem to originate from $P_2$.
   ii) Parabolic mirror
   - Parallel rays converge in the focal point (focal length $f$).
   - Applications: Telescope, collimator
iii) Elliptic mirror
- Rays originating from focal point $P_1$ converge in the second focal point $P_2$

iv) Spherical mirror
- Neither imaging like elliptical mirror nor focusing like parabolic mirror
- Parallel rays cross the optical axis at different points
- Connecting line of intersections of rays → \textit{caustic}

- Parallel, paraxial rays converge to the focal point $f = (-R)/2$
- Convention: $R < 0$ - concave mirror; $R > 0$ - convex mirror.
- For paraxial rays the spherical mirror acts as a focusing as well as an imaging optical element. Paraxial rays emitted in point $P_1$ are reflected and converge in point $P_2$

\begin{align*}
\frac{1}{z_1} + \frac{1}{z_2} &= \frac{2}{(-R)} \\
\text{(imaging formula)}
\end{align*}

\text{Paraxial imaging: imaging formula and magnification} \\
\[ m = -\frac{z_2}{z_1} \] (proof given in exercises)

B) Planar interface
- Snell's law: $n_1 \sin \theta_1 = n_2 \sin \theta_2$
- For paraxial rays: $n_1 \theta_1 = n_2 \theta_2$
- External reflection ($n_1 < n_2$): ray refracted away from the interface
- Internal reflection ($n_1 > n_2$): ray refracted towards the interface
- Total internal reflection (TIR) for:

\[ \theta_2 = \frac{\pi}{2} \Rightarrow \sin \theta_1 = \sin \theta_{\text{TIR}} = \frac{n_2}{n_1} \]

C) Spherical interface (paraxial)
- Paraxial imaging
\[ \theta_2 \approx \frac{n_1}{n_2} - \frac{n_2 - n_1}{n_2} \frac{y}{R} \quad (\star) \]

\[ m = -\frac{n_1}{n_2} \frac{z_1}{z_1} \quad \text{(magnification)} \]

(Proof: exercise)

- if paraxiality is violated \( \rightarrow \) aberration
- rays coming from one point of the object do not intersect in one point of the image (caustic)

D) Spherical thin lens (paraxial)

\[ \frac{n_1}{z_1} \approx \frac{n_2 - n_1}{R} \quad \text{(imaging formula)} \]

\[ m = -\frac{n_1}{n_2} \frac{z_1}{z_1} \quad \text{(magnification)} \]

\[ \frac{1}{z_1} + \frac{1}{z_2} \approx \frac{1}{f} \quad \text{(imaging formula)} \]

\[ m = -\frac{z_2}{z_1} \quad \text{(magnification)} \]

(compare to spherical mirror)

1.5 Ray tracing in inhomogeneous media (graded-index - GRIN optics)

- \( n(r) \) - continuous function, fabricated by, e.g., doping
- curved trajectories \( \rightarrow \) graded-index layer can act as, e.g., a lens

1.5.1 Ray equation

Starting point: we minimize the optical path or the delay (Fermat)

\[ \delta \int_A^B n(r) ds = 0 \]

computation:

\[ L = \int_A^B n[r(s)] ds \]

variation of the path: \( r(s) + \delta r(s) \)
\[\delta L = \int_a^n \delta n \, ds + \int_a^n n \delta ds\]

\[\delta n = \text{grad} n \cdot \delta \mathbf{r}\]

\[\delta ds = \sqrt{(dr + d\delta \mathbf{r})^2} - \sqrt{(dr)^2} = \sqrt{(dr)^2 + 2dr \cdot d\delta \mathbf{r} + (d\delta \mathbf{r})^2} - \sqrt{(dr)^2} \approx ds \sqrt{1 + 2 \frac{dr}{ds} \cdot \frac{d\delta \mathbf{r}}{ds} + \left(\frac{d\delta \mathbf{r}}{ds}\right)^2} - ds \approx ds \left(1 + \frac{dr}{ds} \cdot \frac{d\delta \mathbf{r}}{ds}\right) - ds = ds \frac{dr}{ds} \frac{d\delta \mathbf{r}}{ds}\]

\[\delta L = \int_a^n \left(\text{grad} n \cdot \delta \mathbf{r} + n \frac{dr}{ds} \frac{d\delta \mathbf{r}}{ds}\right) ds\]

\[= \int_a^n \left(\text{grad} n - \frac{d}{ds} \left(n \frac{dr}{ds}\right)\right) \delta \mathbf{r} ds\]

\[\delta L = 0 \text{ for arbitrary variation}\]

\[\text{grad} n = \frac{d}{ds} \left(n \frac{dr}{ds}\right)\] ray equation

**Possible solutions:**

**A)** trajectory

\[- x(z), y(z) \text{ and } ds = dz \sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2}\]

- solve for \(x(z), y(z)\)
- paraxial rays \(\rightarrow (ds \approx dz)\)

\[\frac{d}{dz} \left[ n(x, y, z) \frac{dx}{dz}\right] \approx \frac{dn}{dx}\]

\[\frac{d}{dz} \left[ n(x, y, z) \frac{dy}{dz}\right] \approx \frac{dn}{dy}\]

**B)** homogeneous media

- straight lines

**C)** graded-index layer \(n(y)\) - paraxial, SELFOC

\[\text{paraxial } \rightarrow \frac{dy}{dz} \ll 1 \text{ and } dz \approx ds\]

\[n^2(y) = n_0^2 \left(1 - \alpha^2 y^2\right) \Rightarrow n(y) = n_0 \left(1 - \frac{1}{2} \alpha^2 y^2\right) \text{ for } \alpha \ll 1\]

\[\frac{d}{ds} \left[ n(y) \frac{dy}{ds}\right] \approx \frac{d}{dz} \left[ n(y) \frac{dy}{dz}\right] \approx \frac{d^2 y}{ds^2} \Rightarrow \frac{d^2 y}{dz^2} = 1 \frac{dn(y)}{n(y) dy}\]

\[y(z) = y_0 \cos \alpha z + \frac{\theta_0}{\alpha} \sin \alpha z\]

\[\theta(z) = \frac{dy}{dz} = -y_0 \alpha \sin \alpha z + \theta_0 \cos \alpha z\]

\[\text{for } n(y)-n_0 < 1: \frac{d^2 y}{dz^2} = -\alpha^2 y \rightarrow \frac{dy}{dz} = -y_0 \alpha \sin \alpha z + \theta_0 \cos \alpha z\]

**1.5.2 The eikonal equation**

- bridge between geometrical optics and wave
- eikonal \(S(r) = \text{constant} \rightarrow \) planes perpendicular to rays
- from \(S(r)\) we can determine direction of rays \(\sim \text{grad} S(r)\) (like potential)

\[\left[\text{grad} S(r)\right]^2 = n(r)^2\]

**Remark:** it is possible to derive Fermat’s principle from eikonal equation

- geometrical optics: Fermat’s or eikonal equation

\[S(r_b) - S(r_a) = \int_{r_a}^{r_b} \text{grad} S(r) \, ds = \int_a^n n(r) \, ds\]

eikonal \(\rightarrow\) optical path length \(\sim\) phase of the wave

**1.6 Matrix optics**

- technique for paraxial ray tracing through optical systems
- propagation in a single plane only
- rays are characterized by the distance to the optical axis \((y)\) and their inclination \((\theta)\) \(\rightarrow\) two algebraic equation \(\rightarrow 2 \times 2\) matrix

Advantage: we can trace a ray through an optical system of many elements by multiplication of matrices.
1.6.1 The ray-transfer-matrix

In paraxial approximation:

\[ y_2 = Ay_1 + B\theta_1 \]
\[ \theta_2 = Cy_1 + D\theta_1 \]

\( A=0 \): same \( \theta_1 \) \( \rightarrow \) same \( y_2 \) \( \rightarrow \) focusing

\( D=0 \): same \( y_1 \) \( \rightarrow \) same \( \theta_2 \) \( \rightarrow \) collimation

1.6.2 Matrices of optical elements

A) free space

\[ M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \]

B) refraction on planar interface

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{bmatrix} \]

C) refraction on spherical interface

D) thin lens

\[ M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \]

E) reflection on planar mirror

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

F) reflection on spherical mirror (compare to lens)

\[ M = \begin{bmatrix} 1 & 0 \\ 2/R & 1 \end{bmatrix} \]

1.6.3 Cascaded elements

\[ y_{N+1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} y_1 \rightarrow M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

\[ \theta_{N+1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \theta_1 \rightarrow M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

\( M = M_N \ldots M_2 M \)
2. Optical fields in dispersive and isotropic media

2.1 Maxwell’s equations

Our general starting point is the set of Maxwell’s equations. They are the basis of the electromagnetic approach to optics which is developed in this lecture.

2.1.1 Adaption to optics

The notation of Maxwell’s equations is different for different disciplines of science and engineering which rely on these equations to describe electromagnet phenomena at different frequency ranges. Even though Maxwell’s equations are valid for all frequencies, the physics of light matter interaction is different for different frequencies. Since light matter interaction must be included in the Maxwell’s equations to solve them consistently, different ways have been established how to write down Maxwell’s equations for different frequency ranges. Here we follow a notation which was established for a convenient notation at frequencies close to visible light.

Maxwell’s equations (macroscopic)

In a rigorous way the electromagnetic theory is developed starting from the properties of electromagnetic fields in vacuum. In vacuum one could write down Maxwell’s equations in their so-called pure microscopic form, which includes the interaction with any kind of matter based on the consideration of point charges. Obviously this is inadequate for the description of light in condensed matter, since the number of point charges which would need to be taken into account to describe a macroscopic object, would exceed all imaginable computational resources.

To solve this problem one uses an averaging procedure, which summarizes to influence of many point charges on the electromagnetic field in a homogeneously distributed response of the solid state on the excitation by the light. In turn, also the electromagnetic fields are averaged over some adequate volume. For optics this procedure is justified, since any kind of available experimental detector could not resolve the very fine spatial details of the fields in between the point charges, e.g. ions or electrons, which are lost by this averaging.

These averaged electromagnetic equations have been rigorously derived in a number of fundamental text books on electro-dynamic theory. Here we will not redo this derivation. We will rather start directly from the averaged Maxwell’s equations equation.

\[
\begin{align*}
\text{rot} \mathbf{E}(r,t) &= -\frac{\partial \mathbf{B}(r,t)}{\partial t} \\
\text{div} \mathbf{D}(r,t) &= \rho_{\text{ext}}(r,t) \\
\text{rot} \mathbf{H}(r,t) &= \mathbf{j}_{\text{macro}}(r,t) + \frac{\partial \mathbf{D}(r,t)}{\partial t} \\
\text{div} \mathbf{B}(r,t) &= 0
\end{align*}
\]

- electric field (G: elektrisches Feld) \( \mathbf{E}(r,t) \) \([\text{V/m}]\)
- magnetic flux density (magnetic induction) (G: magnetische Flussdichte oder magnetische Induktion) \( \mathbf{B}(r,t) \) \([\text{Vs/m}^2] \) or [tesla]
- electric flux density (electric displacement field) (G: elektrische Flussdichte oder dielektrische Verschiebung) \( \mathbf{D}(r,t) \) \([\text{As/m}^2]\)
- external charge density \( \rho_{\text{ext}}(r,t) \) \([\text{As/m}^3]\)
- macroscopic current density \( \mathbf{j}_{\text{macro}}(r,t) \) \([\text{A/m}^2]\)

Auxiliary fields

The "cost" of the introduction of macroscopic Maxwell's equations is the occurrence of two additional fields, the dielectric flux density \( \mathbf{D}(r,t) \) and the magnetic field \( \mathbf{H}(r,t) \). These two fields are related to the electric field \( \mathbf{E}(r,t) \) and magnetic flux density \( \mathbf{B}(r,t) \) by two other new fields.

\[
\begin{align*}
\mathbf{D}(r,t) &= \varepsilon_0 \mathbf{E}(r,t) + \mathbf{P}(r,t) \\
\mathbf{H}(r,t) &= \frac{1}{\mu_0} \left[ \mathbf{B}(r,t) - \mathbf{M}(r,t) \right]
\end{align*}
\]

- dielectric polarization (G: dielektrische Polarisierung) \( \mathbf{P}(r,t) \) \([\text{As/m}^2]\),
- magnetic polarization (magnetization) (G: Magnetisierung) \( \mathbf{M}(r,t) \) \([\text{Vs/m}^2]\)
- electric constant (vacuum permittivity) (G: Vakuumpermittivität) \( \varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854 \times 10^{-12} \text{ As/Vm} \)
- magnetic constant (vacuum permeability) (G: Vakuumpemereabilität) \( \mu_0 = 4\pi \times 10^{-7} \text{ Vs/Am} \)

Light matter interaction

In order to solve this set of equations, i.e. Maxwell's equations and auxiliary field equations one needs to connect the dielectric flux density \( \mathbf{D}(r,t) \) and the
magnetic field $\mathbf{H}(\mathbf{r},t)$ to the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic flux density $\mathbf{B}(\mathbf{r},t)$. This is achieved by modeling the material properties by introducing the material equations.

- The effect of the medium gives rise to polarization $\mathbf{P}(\mathbf{r},t) = \int \mathbf{E} \, d\mathbf{r}$ and magnetization $\mathbf{M}(\mathbf{r},t) = \int \mathbf{B} \, d\mathbf{r}$. In order to solve Maxwell’s equations we need material models describing these quantities.

- In optics, we generally deal with non-magnetizable media $\implies \mathbf{M}(\mathbf{r},t) = 0$ (exceptions are metamaterials with $\mathbf{M}(\mathbf{r},t) \neq 0$).

Furthermore we need to introduce sources of the fields into our model. This is achieved by the so-called source terms which are inhomogeneities and hence they define unique solutions of the equations.

- free charge density $\rho_{\text{ext}}(\mathbf{r},t)$ [As/m$^3$]

- macroscopic current density $\mathbf{j}_{\text{macro}}(\mathbf{r},t)$ [A/m$^2$]
  
  - conductive current density $\mathbf{j}_{\text{cond}}(\mathbf{r},t) = \int \mathbf{E} \, d\mathbf{r}$
  
  - convective current density $\mathbf{j}_{\text{conv}}(\mathbf{r},t) = \rho_{\text{ext}}(\mathbf{r},t) \mathbf{v}(\mathbf{r},t)$

  In optics, we generally have no free charges which change at speeds comparable to the frequency of light:

  $\rho_{\text{ext}}(\mathbf{r},t) = 0 \implies \mathbf{j}_{\text{conv}}(\mathbf{r},t) = 0$.

With the above simplifications, we can formulate Maxwell’s equations in the context of optics:

\[
\begin{align*}
\text{rot} \mathbf{E}(\mathbf{r},t) &= -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} - \varepsilon_0 \varepsilon_r \mathbf{E}(\mathbf{r},t) = -\mathbf{P}(\mathbf{r},t) \\
\text{rot} \mathbf{H}(\mathbf{r},t) &= \mathbf{j}(\mathbf{r},t) + \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \varepsilon_r \frac{\partial \mathbf{P}(\mathbf{r},t)}{\partial t} \\
&\quad \text{div} \mathbf{H}(\mathbf{r},t) = 0
\end{align*}
\]

- In optics, the medium (or more precisely the mathematical material model) determines the dependence of the polarization on the electric field $\mathbf{P}(\mathbf{E})$ and the dependence of the (conductive) current density on the electric field $\mathbf{j}(\mathbf{E})$.

- Once we have specified these relations, we can solve Maxwell’s equations consistently.

Example:

- In vacuum, both polarization and current density are zero, and we can solve Maxwell’s equations directly (most simple material model).

Remark:

- We can define a bound charge density $\rho_{\text{b}}(\mathbf{r},t) = -\text{div} \mathbf{P}(\mathbf{r},t)$

- and a bound current density $\mathbf{j}_{\text{b}}(\mathbf{r},t) = \frac{\partial \mathbf{P}(\mathbf{r},t)}{\partial t}$

- This essentially means that we can describe the same physics in two different ways (see generalized complex dielectric function below).

Complex field formalism (G: komplexer Feld-Formalismus):

- Maxwell’s equations are also valid for complex fields and are easier to solve.

- This fact can be exploited to simplify calculations, because it is easier to deal with complex exponential functions $\exp(\mathbf{ix})$ than with trigonometric functions $\cos(x)$ and $\sin(x)$.

Convention in this lecture

- real physical field: $\mathbf{E}(\mathbf{r},t)$

- complex mathematical representation: $\mathbf{E}(\mathbf{r},t)$

- They are related by

\[
\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2}} \left[ \mathbf{E}(\mathbf{r},t) + \mathbf{E}^*(\mathbf{r},t) \right] = \text{Re}\left[ \mathbf{E}(\mathbf{r},t) \right]
\]

Remark: This relation can be defined differently in different textbooks.

- This means in general: For calculations we use the complex fields $[\mathbf{E}(\mathbf{r},t)]$ and for physical results we go back to real fields by simply omitting the imaginary part. This works because Maxwell’s equations are linear and no multiplications of fields occur.

- Therefore, be careful when multiplications of fields are required to go back to real quantities before! This is relevant for, e.g., calculation of Poynting vector, see Chapter below.

2.1.2 Temporal dependence of the fields

When it comes to time dependence of the electromagnetic field, we can distinguish two different types of light:

A) monochromatic light $\leftrightarrow$ stationary fields

- harmonic dependence on temporal coordinate
phase is fixed \(\rightarrow\) coherent, infinite wave train e.g.:
\[
E(r, t) = E(r) \exp(-i\omega t)
\]

Monochromatic light approximates very well the typical output of a continuous wave (CW) laser. Once we know the frequency we have to compute the spatial dependence of the (stationary) fields only.

B) polychromatic light \(\leftrightarrow\) non-stationary fields

- finite wave train
- With the help of Fourier transformation we can decompose the fields into infinite wave trains and use all the results from case A) (see next section)

\[
E(r, t) = \int_{-\infty}^{\infty} E(r, \omega) \exp(-i\omega t) d\omega
\]

\[
\bar{E}(r, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(r, t) \exp(i\omega t) dt
\]

Remark: The position of the sign in the exponent and the factor \(1/2\pi\) can be defined differently in different textbooks.

2.1.3 Maxwell’s equations in Fourier domain

We want to plug the Fourier decompositions of our fields into Maxwell’s equations in order to get a more simple description. For this purpose, we need to know how a time derivative transforms into Fourier space. Here we used integration by parts:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \left[ \frac{\partial}{\partial t} E(r, t) \right] \exp(i\omega t) = -i\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ E(r, t) \exp(i\omega t) = -i\omega \bar{E}(r, \omega)
\]

\(\rightarrow\) rule:

\[\frac{\partial}{\partial t} \rightarrow -i\omega\]

Now we can write Maxwell’s equations in Fourier domain:

\[
\rot E(r, \omega) = i\omega \mu_0 H(r, \omega) \quad \quad \quad \quad \quad \varepsilon_0 \div E(r, \omega) = -\div P(r, \omega)
\]

\[
\rot H(r, \omega) = j(r, \omega) - i\omega \mu_0 P(r, \omega) - i\varepsilon_0 \vec{E}(r, \omega) \quad \div H(r, \omega) = 0
\]

2.1.4 From Maxwell’s equations to the wave equation

Maxwell's equations provide the basis to derive all possible mathematical solutions of electromagnetic problems. However very often we are interested just in the radiation fields which can be described more easily by an adapted equation, which is the so-called wave equation. From Maxwell’s equations it is straight forward to derive the wave equation by using the two curl equations.

A) Time domain derivation

We start from applying the curl operator \(\rot\) a second time on \(\rot E(r, t) = \ldots\) and substitute \(\rot H\) with the other Maxwell equation

\[
\rot E(r, t) = -\mu_0 \frac{\partial}{\partial t} \left[ j(r, t) + \frac{\partial P(r, t)}{\partial t} + \varepsilon_0 \frac{\partial E(r, t)}{\partial t} \right]
\]

And find the wave equation for the electric field

\[
\rot E(r, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(r, t) = -\mu_0 \frac{\partial j(r, t)}{\partial t} - \mu_0 \frac{\partial^2 P(r, t)}{\partial t^2}
\]

The blue terms require knowledge of the material model. Additionally, we have to make sure that all other Maxwell’s equations are fulfilled, in particular:

\[
\div \left[ \varepsilon_0 E(r, t) + P(r, t) \right] = 0
\]

Once we have solved the wave equation, we know the electric field. From that we can easily compute the magnetic field:

\[
\frac{\partial H(r, t)}{\partial t} = -\frac{1}{\mu_0} \rot E(r, t)
\]

Remarks:

- An analog procedure is possible for \(H\), i.e., we can derive a wave equation for the magnetic field.
- Generally, the wave equation for \(E\) is more convenient, because the material model defines \(P(E)\).
- However, for inhomogeneous media \(H\) can be the better choice for the numerical solution of the partial differential equation since it forms a hermitian operator.
- analog procedure possible for \(H \rightarrow E\)
- generally, wave equation for \(E\) is more convenient, because \(P(E)\) given
- for inhomogeneous media \(H\) can be better choice

B) Frequency domain derivation

We can do the same procedure in the Fourier domain and find

\[
\rot \vec{E}(r, \omega) - \frac{\omega^2}{c^2} \vec{E}(r, \omega) = \xi_0 \mu_0 j(r, \omega) + \mu_0 \omega^2 \vec{P}(r, \omega)
\]

and

\[
\div \left[ \varepsilon_0 \vec{E}(r, \omega) + \vec{P}(r, \omega) \right] = 0
\]

- magnetic field:
\[ \mathbf{H}(r, \omega) = -\frac{i}{\omega} \mathbf{\nabla} \times \mathbf{E}(r, \omega) \]

- transferring the results from the Fourier domain to the time domain
  - for stationary fields: take solution and multiply by \( e^{i\omega t} \).
  - for non-stationary fields and linear media \( \Rightarrow \) inverse Fourier transformation

\[ \mathbf{E}(r, t) = \int_{-\infty}^{\infty} \mathbf{E}(r, \omega) \exp(-i\omega t) d\omega \]

2.1.5 Decoupling of the vectorial wave equation

So far we have seen that for the general problem of electromagnetic waves all 3 field components of the electric or the magnetic field are coupled. Hence we have to solve a vectorial wave equation for the general problem. However, it would be desirable to express problems also by scalar equation since they are much easier to solve. For problems with translational invariance in at least one direction, as e.g. for homogeneous infinite media, layers or interfaces, this can be achieved since the vectorial components of the fields can be decoupled.

Let’s assume invariance in the \( y \)-direction and propagation only in the \( x-z \)-plane. Then all spatial derivatives along the \( y \)-direction disappear \( \left( \frac{\partial}{\partial y} = 0 \right) \) and the operators in the wave equation simplify.

\[ \mathbf{\nabla} \times \mathbf{E} = \mathbf{\nabla} \times \mathbf{E} = \Delta \mathbf{E} = \begin{bmatrix} \frac{\partial}{\partial z} \left( \frac{\partial \mathbf{E}_z}{\partial x} + \frac{\partial \mathbf{E}_x}{\partial z} \right) \\ 0 \\ \frac{\partial}{\partial x} \left( \frac{\partial \mathbf{E}_x}{\partial z} + \frac{\partial \mathbf{E}_y}{\partial y} \right) \end{bmatrix} \]

The decoupling becomes visible when the three components of the general vectorial field are decomposed in the following way.

- decomposition of electric field

\[ \mathbf{E} = \mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z \]

\[ \mathbf{E} = \begin{bmatrix} 0 \\ \mathbf{E}_x \\ 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{E}_y \\ 0 \\ \mathbf{E}_z \end{bmatrix} \]

with Nabla operator \( \mathbf{V}^{(2)} = \begin{bmatrix} \frac{\partial}{\partial x} \\ 0 \\ \frac{\partial}{\partial z} \end{bmatrix} \), and Laplace \( \Delta^{(2)} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \)

Hence we obtain two wave equations for the \( \mathbf{E}_x \) and \( \mathbf{E}_y \) fields.

- gives two decoupled wave equations

These two wave equations are independent as long as the material response, which is expressed by \( \mathbf{j} \) and \( \mathbf{P} \), does not couple the respective field components by some anisotropic response.

Properties

- propagation of perpendicularly polarized fields \( \mathbf{E}_x \) and \( \mathbf{E}_y \) can be treated separately
- propagation of \( \mathbf{E}_z \) is described by scalar equation
- similarly the other field components can be described by a scalar equation for \( \mathbf{H} \)
- alternative notations:
  - \( \perp \rightarrow s \rightarrow \text{TE (transversal electric)} \)
  - \( \parallel \rightarrow p \rightarrow \text{TM (transversal magnetic)} \)

2.2 Optical properties of matter

In this chapter we will derive a simple material model for the polarization and the current density. The basic idea is to write down an equation of motion for a single exemplary charged particle and assume that all other particles of the same type behave similarly. More precisely, we will use a driven harmonic oscillator model to describe the motion of bound charges giving rise to a polarization of the medium. For free charges we will use the same model but without restoring force, leading eventually to a current density. In the literature, this simple approach is often called the Drude-Lorentz model (named after Paul Drude and Hendrik Antoon Lorentz).

2.2.1 Basics

We are looking for \( \mathbf{P}(\mathbf{E}) \) and \( \mathbf{j}(\mathbf{E}) \). In general, this leads to a many body problem in solid state theory which is rather complex. However, in many cases phenomenological models are sufficient to describe the necessary phenomena. As already pointed out above, we use the simplest approach, the so-called Drude-Lorentz model for free or bound charge carriers (electrons).

- assume an ensemble of non-coupling, driven, and damped harmonic oscillators
- free charge carriers: metals and excited semiconductors (intraband)
- bound charge carriers: dielectric media and semiconductors (interband)
The Drude-Lorentz model creates a link between cause (electric field) and effect (induced polarization or current). Because the resulting relations $P(E)$ and $j(E)$ are linear (no $E^2$ etc.), we can use linear response theory.

For the polarization $P(E)$ (for $j(E)$ very similar):
- description in both time and frequency domain possible
- **In time domain:** we introduce the response function ($G$: Responsfunktion)
  $$P(r,t) = \varepsilon_0 \sum_j \hat{R}_j(t-t') E_j(r,t')dt'$$
  with $\hat{R}$ being a 2nd rank tensor
  $i=x,y,z$ and summing over $j=x,y,z$
- **In frequency domain:** we introduce the susceptibility ($G$: Suszeptibilität)
  $$P(r,\omega) = \varepsilon_0 \sum_j \chi_{\omega}(r,\omega) E_j(r,\omega)$$
  - response function and susceptibility are linked via Fourier transform (convolution theorem)
  $$\hat{R}_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_{\omega}(r,\omega) \exp(-i\omega t) d\omega$$
- Obviously, things look friendlier in frequency domain. Using the wave equation from before and assuming that there are no currents ($j=0$) we find
  $$\text{rotrot } \mathbf{E}(r,\omega) - \frac{\omega^2}{c^2} \mathbf{E}(r,\omega) = \mu_0 \omega^2 \mathbf{P}(r,\omega)$$
  or
  $$\nabla^2 \mathbf{E}(r,\omega) + \frac{\omega^2}{c^2} \mathbf{E}(r,\omega) - \nabla \cdot \mathbf{D}(r,\omega) = -\mu_0 \omega^2 \mathbf{P}(r,\omega)$$
- and for auxiliary fields
  $$\mathbf{D}(r,\omega) = \varepsilon_0 \mathbf{E}(r,\omega) + \mathbf{P}(r,\omega)$$

The general response function and the respective susceptibility given above simplifies for certain properties of the medium:

- **Simplification of the wave equation for different types of media**
  A) linear, homogenous, isotropic, non-dispersive media (most simple but very unphysical case)
  - homogenous $\rightarrow \chi_{\omega}(r,\omega) = \chi_{\omega}(\omega)$
  - isotropic $\rightarrow \chi_{\omega}(r,\omega) = \chi(\omega)\delta_\omega$
  - non-dispersive $\rightarrow \chi(\omega) = \chi(\omega) \rightarrow$ instantaneous: $\hat{R}_j(r,t) = \chi_{\omega}(r) \delta(t)$
  (Attention: This is unphysical!)
  $\mathbf{\chi}(\omega) \rightarrow \mathbf{\chi}$ is a scalar constant
  - approximation is valid only for a certain frequency range, because all media are dispersive
  - based on an unphysical material model

B) linear, homogeneous, isotropic, **dispersive** media $\rightarrow \chi(\omega)$
  - Maxwell: $\text{div } \mathbf{D}(r,\omega) = 0 \Rightarrow \text{div } \mathbf{E}(r,\omega) = 0$ for $\varepsilon(\omega) \neq 0$
  $$\Delta \mathbf{E}(r,\omega) + \frac{\omega^2}{c^2} \mathbf{E}(r,\omega) = 0 \rightarrow \Delta \mathbf{E}(r,\omega) - \varepsilon \frac{\omega^2}{c^2} \mathbf{E}(r,\omega) = 0$$
  Helmholtz equation
  - This description is sufficient for many materials.

C) linear, **inhomogeneous**, isotropic, dispersive media $\rightarrow \chi(r,\omega)$
  - Maxwell: $\text{div } \mathbf{D}(r,\omega) = 0 \land \text{div } \mathbf{E}(r,\omega) = 0$ for $\varepsilon(\omega) \neq 0$
  $$\Delta \mathbf{E}(r,\omega) + \frac{\omega^2}{c^2} \mathbf{E}(r,\omega) = 0$$
  - and for auxiliary fields
  $$\mathbf{D}(r,\omega) = \varepsilon(r) \mathbf{E}(r,\omega) + \varepsilon_0 \mathbf{E}(r,\omega)$$
  $$\text{div } \mathbf{D}(r,\omega) = 0$$
  $$\text{div } \mathbf{D}(r,\omega) = \varepsilon_0 \varepsilon(\omega) \text{div } \mathbf{E}(r,\omega) + \varepsilon_0 \varepsilon(\omega) \text{grad } \varepsilon(r,\omega) = 0$$
  $$\Rightarrow \text{div } \varepsilon(\omega) \mathbf{E}(r,\omega) = -\text{grad } \varepsilon(r,\omega) \cdot \mathbf{E}(r,\omega).$$
Before we start writing down the actual material model equations, let us summarize what we want to do:

**What kind of light-matter interaction do we want to consider?**

**I) Interaction of light with bound electrons and the lattice**

The contributions of bound electrons and lattice vibrations in dielectrics and semiconductors give rise to the polarization \( \mathbf{P} \). The lattice vibrations (phonons) are the ionic part of the material model. Because of the large mass of the ions \( (10^3 \times \text{mass of electron}) \) the resulting oscillation frequencies will be small. Generally speaking, phonons are responsible for thermal properties of the medium. However, some phonon modes may contribute to optical properties, but they have small dispersion (weak dependence on frequency \( \omega \)).

Fully understanding the electronic transitions of bound electrons requires quantum theoretical treatment, which allows an accurate computation of the transition frequencies. However, a (phenomenological) classical treatment of the oscillation of bound electrons is possible and useful.

**II) Interaction of light with free electrons**

The contribution of free electrons in metals and excited semiconductors gives rise to a current density \( \mathbf{j} \). We assume a so-called (interaction-)free electron gas, where the electron charges are neutralized by the background ions. Only collisions with ions and related damping of the electron motion will be considered.

We will look at the contributions from I) and II) separately, and join the results later.

**2.2.2 Dielectric polarization and susceptibility**

Let us first focus on bound charges (ions, electrons). In the so-called Drude model, the electric field \( \mathbf{E}(r, \omega) \) gives rise to a displacement \( \mathbf{s}(r, \omega) \) of charged particles from their equilibrium positions. In the easiest approach this can be modeled by a driven harmonic oscillator:

\[
\frac{\partial^2}{\partial t^2} \mathbf{s}(r, t) + g \frac{\partial}{\partial t} \mathbf{s}(r, t) + \omega_0^2 \mathbf{s}(r, t) = \frac{q}{m} \mathbf{E}(r, t)
\]

- resonance frequency (electronic transition) \( \Rightarrow \omega_0 \)
- damping \( \Rightarrow g \)
- charge \( \Rightarrow q \)
- mass \( \Rightarrow m \)

The induced electric dipole moment due to the displacement of charged particles is given by

\[
\mathbf{p}(r, t) = q \mathbf{s}(r, t),
\]

We further assume that all bound charges of the same type behave identical, i.e., we treat an ensemble of non-coupled, driven, and damped harmonic oscillators. Then, the dipole density (polarization) is given by

\[
\mathbf{P}(r, t) = N \mathbf{p}(r, t) = Nq \mathbf{s}(r, t)
\]

Hence, the governing equation for the polarization \( \mathbf{P}(r, t) \) reads as

\[
\frac{\partial^2}{\partial t^2} \mathbf{P}(r, t) + \frac{\partial}{\partial t} \mathbf{P}(r, t) + \omega_0^2 \mathbf{P}(r, t) = \frac{q^2 N}{m} \mathbf{E}(r, t) = \varepsilon_0 \varepsilon \mathbf{E}(r, t)
\]

with oscillator strength \( f = \frac{1}{\varepsilon_0} \frac{e^2 N}{m} \), for \( q = -e \) (electrons).

This equation is easy to solve in Fourier domain:

\[
-\omega^2 \mathbf{P}(r, \omega) - ig\omega \mathbf{P}(r, \omega) + \omega_0^2 \mathbf{P}(r, \omega) = \varepsilon_0 \frac{f}{\omega_0^2 - \omega^2 - ig\omega} \mathbf{E}(r, \omega)
\]

\[
\mathbf{P}(r, \omega) = \varepsilon_0 \frac{f}{\omega_0^2 - \omega^2 - ig\omega} \mathbf{E}(r, \omega)
\]

with \( \mathbf{P}(r, \omega) = \varepsilon_0 \chi(\omega) \mathbf{E}(r, \omega) \Rightarrow \chi(\omega) = \frac{f}{\omega_0^2 - \omega^2 - ig\omega} \).

In general we have several different types of oscillators in a medium, i.e., several different resonance frequencies. Nevertheless, since in a good approximation they do not influence each other, all these different oscillators contribute individually to the polarization. Hence the model can be constructed by simply summing up all contributions.

- several resonance frequencies

\[
\mathbf{P}(r, \omega) = \varepsilon_0 \sum_j \left( \frac{f_j}{\omega_0^2_j - \omega^2 - ig\omega_j} \right) \mathbf{E}(r, \omega) = \varepsilon_0 \chi(\omega) \mathbf{E}(r, \omega)
\]
\[ \chi(\omega) = \sum_j \left[ \frac{f_j}{(\omega_0 - j\omega^2)} - ig_j \omega \right] \]

- \( \chi(\omega) \) is the complex, frequency dependent susceptibility
- \( \bar{D}(r, \omega) = \varepsilon_0 \bar{E}(r, \omega) + \varepsilon_0 \chi(\omega) \bar{E}(r, \omega) = \varepsilon_0 \chi(\omega) \bar{E}(r, \omega) \)
- \( \varepsilon(\omega) \) is the complex frequency dependent dielectric function

Example: (plotted for \( \eta \) and \( \kappa \) with \( \varepsilon(\omega) = \left[ \eta(\omega) + i\kappa(\omega) \right] \))

2.2.3 Conductive current and conductivity

Let us now describe the response of a free electron gas with positively charged background (no interaction). Again we use the model of a driven harmonic oscillator, but this time with resonance frequency \( \omega_0 = 0 \). This corresponds to the case of zero restoring force.

\[ \frac{\partial^2}{\partial t^2} s(r, t) + \frac{g}{\omega_0} s(r, t) = -\frac{\varepsilon_0}{m} E(r, t), \]

The resulting induced current density is given by

\[ j(r, t) = -Ne \frac{\partial}{\partial t} s(r, t) \]

and the governing dynamic equation reads as

\[ \frac{\partial}{\partial t} j(r, t) + gj(r, t) = \frac{e^2 N}{m} E(r, t) = \varepsilon_0 \omega_p^2 E(r, t) \]

with plasma frequency \( \omega_p^2 = \frac{1}{\varepsilon_0} \frac{e^2 N}{m} \)

Again we solve this equation in Fourier domain:

\[ -i\omega \tilde{j}(r, \omega) + g\tilde{j}(r, \omega) = \varepsilon_0 \omega_p^2 \tilde{E}(r, \omega) \]

\[ \tilde{j}(r, \omega) = \frac{\varepsilon_0 \omega_p^2}{g - i\omega} \tilde{E}(r, \omega) = \sigma(\omega) \tilde{E}(r, \omega). \]

Here we introduced the complex frequency dependent conductivity

\[ \sigma(\omega) = \frac{\varepsilon_0 \omega_p^2}{g - i\omega} = -\frac{i}{g - 1\omega} \frac{\varepsilon_0 \omega_p^2}{\omega - 1\omega} \]

Remarks on plasma frequency

We consider a cloud of electrons and positive ions described by the total charge density \( \rho \) in their self-consistent field \( E \). Then we find according to Maxwell:

\[ \varepsilon_0 \text{div} E(r, t) = \rho(r, t) \]

For cold electrons, and because the total charge is zero, we can use our damped oscillator model from before to describe the current density (only electrons move):

\[ \frac{\partial}{\partial t} j + gj = \varepsilon_0 \omega_p^2 E(r, t) \]

Now we apply divergence operator and plug in from above (red terms):

\[ \text{div} \frac{\partial}{\partial t} j + g\text{div} j = \varepsilon_0 \omega_p^2 \text{div} E(r, t) = \omega_p^2 \rho(r, t) \]

With the continuity equation for the charge density (from Maxwell's equations)

\[ \frac{\partial}{\partial t} \rho + \text{div} j = 0, \]

We can substitute the divergence of the current density and find:

\[ \frac{\partial^2}{\partial t^2} \rho - g \frac{\partial}{\partial t}\rho = \omega_p^2 \rho \]

\[ \frac{\partial^2}{\partial t^2} \rho + g \frac{\partial}{\partial t} \rho + \omega_p^2 = 0 \]

\( \Rightarrow \) harmonic oscillator equation

Hence, the plasma frequency \( \omega_p \) is the eigen-frequency of such a charge density.

2.2.4 The generalized complex dielectric function

In the sections above we have derived expressions for both polarization (bound charges) and conductive current density (free charges). Let us now plug our \( \tilde{j}(r, \omega) \) and \( \tilde{P}(r, \omega) \) into the wave equation (in Fourier domain)
rotrot \( \mathbf{E}(r,\omega) - \frac{\omega^2}{c^2} \mathbf{E}(r,\omega) = \mu_0 \mathbf{\mu}(r,\omega) + i \omega \mu_0 \mathbf{j}(r,\omega) \)

\[ = \left[ \mu_0 \varepsilon_0 \chi(\omega) + i \omega \mu_0 \sigma(\omega) \right] \mathbf{E}(r,\omega) \]

Hence we can collect all terms proportional to \( \varepsilon(\omega) \) and write

rotrot \( \mathbf{E}(r,\omega) = \frac{\omega^2}{c^2} \left[ 1 + \chi(\omega) + \frac{1}{\varepsilon_0} \sigma(\omega) \right] \mathbf{E}(r,\omega) \)

rotrot \( \mathbf{E}(r,\omega) = \frac{\omega^2}{c^2} \varepsilon(\omega) \mathbf{E}(r,\omega) \)

Here, we introduced the generalized complex dielectric function

\[ \varepsilon(\omega) = 1 + \chi(\omega) + \frac{1}{\varepsilon_0} \sigma(\omega) = \varepsilon'(\omega) + i \varepsilon''(\omega) \]

So, in general we have

\[ \varepsilon(\omega) = 1 + \sum_j \left[ \frac{f_j}{\left( \omega_j^2 - \omega^2 \right) - i g_j \omega} \right] + \frac{\omega_0^2}{-\omega^2 - i g_0 \omega} \]

because (from before)

\[ \chi(\omega) = \sum_j \left[ \frac{f_j}{\left( \omega_j^2 - \omega^2 \right) - i g_j \omega} \right], \quad \sigma(\omega) = -\frac{\omega_0}{\omega^2 - i g_0 \omega}. \]

\( \varepsilon(\omega) \) contains contributions from vacuum, phonons (lattice vibrations), bound and free electrons.

**Some special cases for materials in the infrared and visible spectral range:**

**A) Dielectrics (insulators) in the infrared (IR) spectral range near phonon resonance**

If we are interested in dielectrics (insulators) near phonon resonance in the infrared spectral range we can simplify the dielectric function as follows:

\[ \varepsilon(\omega) = 1 + \sum_j \left[ \frac{f_j}{\left( \omega_j^2 - \omega^2 \right) - i g_j \omega} \right] + \frac{f}{\left( \omega_0^2 - \omega^2 \right) - i g_0 \omega} \]

with \( \omega_0 \ll \omega_j \) and \( \omega \sim \omega_0 \)

\[ \Rightarrow \varepsilon(\omega) = \varepsilon_\infty + \frac{f}{\left( \omega_0^2 - \omega^2 \right) - i g_0 \omega} \]

The contribution from electronic transitions shows almost no frequency dependence (dispersion) in this frequency range far away from the electronic resonances. Hence it can be expressed together with the vacuum contribution as a constant \( \varepsilon_\infty \).

Let us study the real and the imaginary part of the resulting \( \varepsilon(\omega) \) separately:

\[ \varepsilon_\infty \rightarrow \text{vacuum and electronic transitions} \]

\[ \varepsilon(\omega) = \Re \varepsilon(\omega) + i \Im \varepsilon(\omega) = \varepsilon'(\omega) + i \varepsilon''(\omega) \]

\[ \varepsilon'(\omega) = \varepsilon_\infty + \frac{f}{\left( \omega_0^2 - \omega^2 \right) + g^2 \omega^2} \]

\[ \varepsilon''(\omega) = \frac{g f \omega}{\left( \omega_0^2 - \omega^2 \right) + g^2 \omega^2} \rightarrow \text{Lorentz curve} \]

**Properties:**

- Resonance frequency: \( \omega_0 \)
- Width of resonance peak: \( g \)
- Static dielectric constant in the limit \( \omega \to 0 \): \( \varepsilon_0 = \varepsilon_\infty + \frac{f}{\omega_0^2} \)
- So called longitudinal frequency \( \omega_L : \varepsilon'(\omega = \omega_L) = 0 \)
- \( \varepsilon'(\omega) \neq 0 \) : absorption and dispersion appear always together
- Near resonance we find \( \varepsilon'(\omega) \ll 0 \) (damping, i.e. decay of field, without absorption if \( \varepsilon'' \approx 0 \))
- Frequency range with normal dispersion: \( \frac{\partial \varepsilon'(\omega)}{\partial \omega} > 0 \)
- Frequency range with anomalous dispersion: \( \frac{\partial^2 \varepsilon'(\omega)}{\partial \omega^2} < 0 \)

**Simplified example:** Sharp resonance for undamped oscillator \( g \to 0 \)
- relation between resonance frequency $\omega_r$ and longitudinal frequency $\omega_L$ (Lyddane-Sachs-Teller relation)

$$
\epsilon' (\omega_r) = \epsilon_\infty + \frac{f}{(\omega_r^2 - \omega_L^2)} = 0, f = (\epsilon_0 - \epsilon_\infty) \omega_0^2 
$$
(from above)

$$
\omega_L = \omega_0 \sqrt{\frac{\epsilon_\infty}{\epsilon_0}}. 
$$

B) Dielectrics in the visible (VIS) spectral range

Dielectric media in visible (VIS) spectral range can be described by a so-called double resonance model, where a phonon resonance exists in the infrared (IR) and an electronic transition exists in the ultraviolet (UV).

$$
\epsilon (\omega) = \epsilon_\infty + \frac{f_p}{(\omega_0^2 - \omega_P^2)} - \frac{f_e}{(\omega_0^2 - \omega_e^2)} \quad \text{with} \quad \omega_0 \ll \omega \ll \omega_0. 
$$

$\epsilon_\infty \rightarrow$ contribution of vacuum and other (far away) resonances

The generalization of this approach in the transparent spectral range leads to the so-called Sellmeier formula.

$$
\epsilon'(\omega) = 1 = \sum_j \frac{\rho_j \omega_j^2}{(\omega_j^2 - \omega^2)}, 
$$
with $j$ being the number of resonances taken into account

- describes many media very well (dispersion of absorption is neglected)
- oscillator strengths and resonance frequencies are often fit parameters to match experimental data

C) Metals in the visible spectral range

If we want to describe metals in the visible spectral range we find

$$
\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + g^2}, \quad \epsilon'(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + g^2}, 
$$
with $\omega_p >> \omega$.

Metals show a large negative real part of the dielectric function $\epsilon'(\omega)$ which gives rise to decay of the fields. Eventually this results in reflection of light at metallic surfaces.

2.2.5 Material models in time domain

Let us now transform our results of the material models back to time domain. In Fourier domain we found for homogeneous and isotropic media:

$$
\tilde{D}(r, \omega) = \epsilon_0 \epsilon(\omega) \tilde{E}(r, \omega), 
\tilde{P}(r, \omega) = \epsilon_0 \chi(\omega) \tilde{E}(r, \omega). 
$$

The response function (or Green's function) $R(t)$ in the time domain is then given by

$$
R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\omega) \exp(-i\omega t) d\omega \quad \chi(\omega) = \int_{-\infty}^{\infty} R(t) \exp(i\omega t) dt. 
$$
To prove this, we can use the convolution theorem

\[ P(r, t) = \int_{-\infty}^{\infty} P(r, \omega) \exp(-i\omega t) d\omega = \varepsilon_0 \int_{-\infty}^{\infty} \chi(\omega) E(r, \omega) \exp(-i\omega t) d\omega \]

\[ = \varepsilon_0 \int_{-\infty}^{\infty} \chi(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} E(r, \omega') \exp(i\omega t') d\omega' \exp(-i\omega t) d\omega \]

Now we switch the order of integration, and identify the response function \( R \) (red terms):

\[ \int_{-\infty}^{\infty} \frac{1}{2\pi} \chi(\omega) \exp(-i\omega(t-t')) d\omega \int_{-\infty}^{\infty} E(r, \omega') d\omega' \]

\[ = \varepsilon_0 \int_{-\infty}^{\infty} R(t-t') E(r, \omega') d\omega' \]

For a “delta” excitation in the electric field we find the response or Greens function as the polarization:

\[ E(r, t') = e^{-i\delta(t'-t_0)} \rightarrow P(r, t) = \varepsilon_0 R(t-t_0)e^{-i\delta(t)} \rightarrow \text{Green’s function} \]

**Examples**

A) instantaneous media (unphysical simplification)

- For instantaneous (or non-dispersive) media, which cannot not really exist in nature, we would find:
  \[ R(t) = \chi(t) \rightarrow P(t) = \varepsilon_0 \chi E(t) \text{ (unphysical!)} \]

B) dielectrics

\[ R_{\mu}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\omega) \exp(-i\omega t) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f}{\varepsilon_0 - \omega^2 - i\omega \gamma} \exp(-i\omega t) d\omega, \]

- Using the residual theorem we find:
  \[ R(t) = \left( \begin{array}{c} f \exp \left( -\frac{g}{2} t \right) \sin \Omega t & \Omega = \sqrt{\frac{1}{\varepsilon_0} - \frac{g^2}{4}} \end{array} \right) \]

\[ P(r, t) = \frac{f}{\Omega} \int_{-\infty}^{\infty} \exp \left( -\frac{g}{2} (t-t') \right) \sin \left( \Omega(t-t') \right) E(r, \omega') d\omega' \]

C) metals

\[ R_{\sigma}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma(\omega) \exp(-i\omega t) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_0 \omega^2}{g} \exp(-i\omega t) d\omega, \]

- Using again the residual theorem we find:

\[ R(t) = \left( \begin{array}{c} \frac{e_0 \omega^2}{g} \exp(-gt) \quad t \geq 0 \\
0 \quad t < 0 \end{array} \right) \]

\[ j(r, t) = \varepsilon_0 \omega^2 \int_{-\infty}^{\infty} \exp(-gt) E(r, \omega') d\omega' \]

### 2.3 The Poynting vector and energy balance

#### 2.3.1 Time averaged Poynting vector

The energy flux of the electromagnetic field is given by the Poynting vector \( S \). In practice, we always measure the energy flux through a surface (detector), \( S \cdot n \), where \( n \) is the normal vector of surface. To be more precise, the Poynting vector \( S(r, t) = \mathbf{E}(r, t) \times \mathbf{H}_r(r, t) \) gives the momentary energy flux. Note that we have to use the real electric and magnetic fields, because a product of fields occurs.

In optics we have to consider the following time scales:

- optical cycle: \( T_o = 2\pi / \omega_o \leq 10^{-14} \text{s} \)
- pulse duration: \( T_p \) in general \( T_p >> T_o \)
- duration of measurement: \( T_m \) in general \( T_m >> T_o \)

Hence, in general the detector does not recognize the fast oscillations of the optical field \( \sim e^{-i\omega t} \) (optical cycles) and delivers a time averaged value. For the situation described above, the electro-magnetic fields factorize in slowly varying envelopes and fast carrier oscillations:

\[ \text{X} \rightarrow \text{X} \]

\[ \frac{1}{2} \mathbf{E}(r, t) \exp(-i\omega t) + c.c. = \mathbf{E}_e(r, t) \]

For such pulses, the momentary Poynting vector reads:
\[
S(r,t) = E_s(r,t) \times H_s(r,t)
\]
\[
= \frac{1}{4} \left[ \vec{E}(r,t) \times \vec{H}^\prime(r,t) + \vec{E}^\prime(r,t) \times \vec{H}(r,t) \right]
\]
\[
+ \frac{1}{4} \left[ \vec{E}(r,t) \times \vec{H}(r,t) \exp\left(-2i\omega_0 t\right) + \vec{E}^\prime(r,t) \times \vec{H}^\prime(r,t) \exp\left(2i\omega_0 t\right) \right]
\]
\[
= \frac{1}{2} \Re \left[ \vec{E}(r,t) \times \vec{H}^\prime(r,t) \right] + \frac{1}{2} \Re \left[ \vec{E}(r,t) \times \vec{H}(r,t) \cos(2\omega_0 t) \right]
\]
\[
+ \frac{1}{2} \Im \left[ \vec{E}^\prime(r,t) \times \vec{H}^\prime(r,t) \sin(2\omega_0 t) \right].
\]

We find that the momentary Poynting vector has some slow contributions which change over time scales of the pulse envelope \(T_p\), and some fast contributions \(-\cos(2\omega_0 t)\), \(-\sin(2\omega_0 t)\) changing over time scales of the optical cycle \(T_0\). Now, a measurement of the Poynting vector over a time interval \(T_m\) leads to a time average of \(S(r,t)\).

\[
\langle S(r,t) \rangle = \frac{1}{T_m} \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} S(r,t') dt' \]

The fast oscillating terms \(-\cos(2\omega_0 t)\) and \(-\sin(2\omega_0 t)\) cancel by the integration since the pulse envelope does not change much over one optical cycle. Hence we get only a contribution from the slow term.

\[
\langle S(r,t) \rangle = \frac{1}{2} \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} \Re \left[ \vec{E}(r,t') \times \vec{H}^\prime(r,t') \right] dt'
\]

Let us now have a look at the special (but important) case of stationary (monochromatic) fields. Then, the pulse envelope does not depend on time at all (infinitely long pulses).

\[
\vec{E}(r,t') = \vec{E}(r), \quad \vec{H}(r,t') = \vec{H}(r)
\]

\[
\langle S(r,t) \rangle = \frac{1}{2} \Re \left[ \vec{E}(r) \times \vec{H}^\prime(r) \right]
\]

This is the definition for the optical intensity \(I = \|S(r,t)\|\). We see that an intensity measurement destroys information on the phase.

\(I = \|S(r,t)\| \rightarrow \) measurement destroys phase information

### 2.3.2 Time averaged energy balance

Let us motivate a little bit further the concept of the Poynting vector. Some interesting insight on the energy flow of light and hence also on the transport of information can be obtained from the Poynting theorem, which is the equation for the energy balance of the electromagnetic field. The Poynting theorem can be derived directly from Maxwell’s equations. We multiply the two curl equations by \(\vec{H}\), resp. \(\vec{E}\) (note that we use real fields):

\[
\vec{H}_r \cdot \text{rot} \vec{E}_r + \mu_0 \vec{H}_r \cdot \frac{\partial}{\partial t} \vec{H}_r = 0
\]

\[-\varepsilon_0 \vec{E}_r \cdot \frac{\partial}{\partial t} \vec{E}_r + \vec{E}_r \cdot \text{rot} \vec{H}_r = \vec{E}_r \cdot (\vec{j} + \frac{\partial}{\partial t} \vec{P})\]

Next, we subtract the two equations and get

\[
\vec{H}_r \cdot \text{rot} \vec{E}_r - \vec{E}_r \cdot \text{rot} \vec{H}_r + \varepsilon_0 \vec{E}_r \cdot \frac{\partial}{\partial t} \vec{E}_r + \mu_0 \vec{H}_r \cdot \frac{\partial}{\partial t} \vec{H}_r = -\vec{E}_r \cdot (\vec{j} + \frac{\partial}{\partial t} \vec{P}).
\]

This equation can be simplified by using the following vector identity:

\[
\text{div}(\vec{E}_r \times \vec{H}_r) = \vec{H}_r \cdot \text{rot} \vec{E}_r - \vec{E}_r \cdot \text{rot} \vec{H}_r
\]

Finally, with \(\vec{E}_r \cdot \frac{\partial}{\partial t} \vec{E}_r = \frac{1}{2} \frac{\partial}{\partial t} E^2_r\) we find Poynting’s theorem

\[
\begin{align*}
\left( \frac{1}{2} \varepsilon_0 \frac{\partial}{\partial t} E^2_r + \frac{1}{2} \mu_0 \frac{\partial}{\partial t} H^2_r + \text{div} (\vec{E}_r \times \vec{H}_r) \right) = -\vec{E}_r \cdot \left( \vec{j} + \frac{\partial}{\partial t} \vec{P} \right)
\end{align*}
\]

This equation has the general form of a balance equation. Here it represents the energy balance. Apart from the appearance of the Poynting vector (energy flux), we can identify the vacuum energy density

\[
u = \varepsilon_0 E^2_r + \frac{1}{2} \mu_0 H^2_r \rightarrow \text{vacuum energy density}
\]

In the case of stationary fields and isotropic media (simple but important)

\[
\begin{align*}
E_s(r,t) &= \frac{1}{2} \left[ \vec{E}(r) \exp(-i\omega_0 t) + \text{c.c.} \right] \\
H_s(r,t) &= \frac{1}{2} \left[ \vec{H}(r) \exp(-i\omega_0 t) + \text{c.c.} \right]
\end{align*}
\]

Time averaging of the left hand side of Poynting’s theorem (*) yields:

\[
\begin{align*}
\left\{ \frac{1}{2} \varepsilon_0 \frac{\partial}{\partial t} E^2_s(r,t) + \frac{1}{2} \mu_0 \frac{\partial}{\partial t} H^2_s(r,t) + \text{div} \left[ E_s(r,t) \times H_s(r,t) \right] \right\} &= \frac{1}{2} \text{div} \left[ \Re \left[ \vec{E}(r) \times \vec{H}^\prime(r) \right] \right] \\
&= \text{div} \langle S(r,t) \rangle.
\end{align*}
\]

Note that the time derivative removes stationary terms in \(E^2_s(r,t)\) and \(H^2_s(r,t)\).

Time averaging of the right hand side of Poynting’s theorem yields (source terms):
Now we use our generalized dielectric function:

\[
\begin{align*}
&= -\frac{1}{4} \left[ -i \omega_0 \varepsilon_0 \left( \varepsilon(\omega_0) + i \frac{\sigma(\omega_0)}{\omega_0 \varepsilon_0} \right) E(r) \exp(-i \omega_0 t) + c.c. \right] E(r) \exp(-i \omega_0 t) + c.c. \\
&= \frac{1}{4} i \omega_0 \varepsilon_0 \left[ \varepsilon(\omega_0) - 1 \right] E(r) E(r)^* + c.c.
\end{align*}
\]

Again, all fast oscillating terms \( \sim \exp(\pm 2i \omega_0 t) \) cancel due to the time average.

Finally, splitting \( \varepsilon(\omega_0) \) into real and imaginary part yields

\[
\begin{align*}
&= \frac{1}{4} i \omega_0 \varepsilon_0 \left[ \varepsilon'(\omega_0) - 1 + i \varepsilon''(\omega_0) \right] E(r) E(r)^* + c.c. = -\frac{1}{2} \omega_0 \varepsilon_0 \varepsilon'(\omega_0) E(r) E'(r).
\end{align*}
\]

Hence, the divergence of the time averaged Poynting vector is related to the imaginary part of the generalized dielectric function:

\[
\text{div} \langle S \rangle = -\frac{1}{2} \omega_0 \varepsilon_0 \varepsilon'(\omega_0) E(r) E'(r).
\]

This shows that a nonzero imaginary part of epsilon \( \varepsilon'(\omega) \neq 0 \) causes a drain of energy flux. In particular, we always have \( \varepsilon'(\omega) > 0 \), otherwise there would be gain of energy. In particular near resonances we have \( \varepsilon'(\omega) \neq 0 \) and therefore absorption.

Further insight into the meaning of \( \text{div} \langle S \rangle \) gives the so-called divergence theorem. If the energy of the electro-magnetic field is flowing through some volume, and we wish to know how much energy flows out of a certain region within that volume, then we need to add up the sources inside the region and subtract the sinks. The energy flux is represented by the (time averaged) Poynting vector, and the Poynting vector's divergence at a given point describes the strength of the source or sink there. So, integrating the Poynting vector's divergence over the interior of the region equals the integral of the Poynting vector over the region's boundary.

\[
\int_V \langle S \rangle dV = \langle S \rangle \cdot n dA
\]
2.4 Normal modes in homogeneous isotropic media

Using the linear material models which we discussed in the previous chapters we can now look for self-consistent solutions to the wave equation include the material response.

It is convenient to use the generalized complex dielectric function for the derivative of the solution of the wave equation

\[ \varepsilon(\omega) = 1 + \chi(\omega) \]

\[ \frac{1}{\omega \varepsilon_0} \sigma(\omega) = \varepsilon'(\omega) + i \varepsilon''(\omega) \]

We will do our analysis in Fourier domain. In particular, we will focus on the most simple solution to the wave equation in Fourier domain, the so-called normal modes. We will see later that it is possible to construct general solutions from the normal modes. The wave equation in Fourier domain reads

\[ \nabla \times (\varepsilon(\omega) \nabla \times \mathbf{E}(\omega, \mathbf{r})) = \mathbf{j}(\omega, \mathbf{r}) \]

According to Maxwell the solutions have to fulfill additionally the divergence equation:

\[ \varepsilon_0 [1 + \chi(\omega)] \nabla \cdot \mathbf{E}(\omega, \mathbf{r}) = 0 \]

In general, this additional condition implies that the electric field is free of divergence:

\[ 1 + \chi(\omega) \neq 0 \rightarrow \nabla \cdot \mathbf{E}(\omega, \mathbf{r}) = 0 \] (normal case)

Let us for a moment assume that we already know that we can find plane wave solutions of the following form in the frequency domain:

\[ \mathbf{E}(\omega, \mathbf{r}) = \mathbf{E}(\omega) \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{r}) \]

\[ \mathbf{k} = \text{unknown complex wave-vector} \]

The corresponding stationary field in time domain is given by:

\[ \mathbf{E}(\mathbf{r}, t) = \mathbf{E} \exp[\mathbf{i} (\mathbf{k} \cdot \mathbf{r} - \omega t)] \]

\[ \Rightarrow \text{monochromatic plane wave} \Rightarrow \text{normal mode} \]

This is a monochromatic plane wave, the simplest solution we can expect, a so-called normal mode.

Then, the divergence condition implies that those waves are transversal:

\[ \mathbf{k} \perp \mathbf{E}(\omega) \Rightarrow \text{transverse wave} \]

If we split the complex wave vector into real and imaginary part \( \mathbf{k} = \mathbf{k}' + \mathbf{i} \mathbf{k}'' \), we can define:

- planes of constant phase \( \mathbf{k}' \mathbf{r} = \text{const.} \)
- planes of constant amplitude \( \mathbf{k}'' \mathbf{r} = \text{const.} \)

In the following we will call the solutions

A) if those planes are identical \( \Rightarrow \text{homogeneous waves} \)

B) if those planes are perpendicular \( \Rightarrow \text{evanescent waves} \)

C) otherwise \( \Rightarrow \text{inhomogeneous waves} \)

We will see that in dielectrics \( (\sigma(\omega) = 0) \) we can find a second, exotic type of wave solutions: At \( \omega = \omega_L \rightarrow \varepsilon(\omega_L) = 0 \), so-called longitudinal waves \( \mathbf{k} \| \mathbf{E}(\omega) \) appear.

2.4.1 Transversal waves

As pointed out above, for \( \omega \neq \omega_L \) the electric field becomes free of divergence:

\[ \varepsilon_0 \varepsilon(\omega) \nabla \times \mathbf{E}(\omega, \mathbf{r}) = \mathbf{j}(\omega, \mathbf{r}) \rightarrow \nabla \times \mathbf{E}(\omega, \mathbf{r}) = \mathbf{0} \]

Then, the wave equation reduces to the Helmholtz equation:

\[ \Delta \mathbf{E}(\omega, \mathbf{r}) + \frac{\omega^2}{c^2} \varepsilon(\omega) \mathbf{E}(\omega, \mathbf{r}) = \mathbf{0} \]

Hence, we have three scalar equations for \( \mathbf{E}(\omega, \mathbf{r}) \) (from Helmholtz), and together with the divergence condition we are left with two independent field components. We will now construct solutions using the plane wave ansatz:

\[ \mathbf{E}(\omega, \mathbf{r}) = \mathbf{E}(\omega) \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{r}) \]

Immediately we see that the wave is transversal:

\[ 0 = \nabla \times \mathbf{E}(\omega, \mathbf{r}) = \mathbf{i} \mathbf{k} \times \mathbf{E}(\omega, \mathbf{r}) \Rightarrow \mathbf{k} \perp \mathbf{E}(\omega) \]

Hence, we have to solve

\[ \left[ -\mathbf{k}^2 + \frac{\omega^2}{c^2} \varepsilon(\omega) \right] \mathbf{E}(\omega) = \mathbf{0} \quad \text{and} \quad \mathbf{k} \cdot \mathbf{E}(\omega) = \mathbf{0}. \]

which leads to the following dispersion relation

\[ \mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \varepsilon(\omega) \]

We see that the so-called wave-number \( \mathbf{k}(\omega) = \frac{\omega}{c} \sqrt{\varepsilon(\omega)} \) is a function of the frequency. We can conclude that transversal plane waves are solutions to Maxwell's equations in homogeneous, isotropic media, only if the dispersion relation \( \mathbf{k}(\omega) \) is fulfilled.

In general, \( \mathbf{k} = \mathbf{k}' + \mathbf{i} \mathbf{k}'' \) is complex. Alternatively it is sometimes useful to introduce the complex refractive index (if \( \mathbf{k}' \| \mathbf{k}'' \)):

\[ k(\omega) = \frac{\omega}{c} \sqrt{\varepsilon(\omega)} = \frac{\omega}{c} n(\omega) = \frac{\omega}{c} [n(\omega) + \mathbf{i} k(\omega)] \]
However, instead of assuming that \( \hat{n}(\omega) \) and \( \sqrt{\varepsilon(\omega)} \) are just the same, one should clearly distinguish between the two. While \( \varepsilon(\omega) \) is a property of the medium, \( \hat{n}(\omega) \) is a property of a particular type of the electromagnetic field in the medium, i.e. a property of the infinitely extended monochromatic plane wave.

\[
\vec{E}(r, \omega) = \vec{E}(\omega) \exp(\text{i}kr)
\]

With the knowledge of the electric field we can compute the magnetic field if desired:

\[
\vec{H}(r, \omega) = -\frac{\text{i}}{\omega \mu_0} \text{rot} \vec{E}(r, \omega) = \frac{1}{\omega \mu_0} \left[ \text{k} \times \vec{E}(\omega) \right] \exp(\text{i}kr)
\]

\[
\rightarrow \vec{H}(r, \omega) = \vec{H}(\omega) \exp(\text{i}kr), \quad \text{with } \vec{H}(\omega) = \frac{1}{\omega \mu_0} \left[ \text{k} \times \vec{E}(\omega) \right]
\]

### 2.4.2 Longitudinal waves

Let us now have a look at the rather exotic case of longitudinal waves. Those waves can only exist for \( \varepsilon(\omega) = 0 \) in dielectrics at the longitudinal frequency \( \omega = \omega_L \). In this case, we cannot conclude that \( \text{div} \vec{E}(r, \omega) = 0 \), and the wave equation reads (the l.h.s. vanishes because \( \varepsilon(\omega) = 0 \)):

\[
\rightarrow \text{rot} \text{rot} \vec{E}(r, \omega_L) = 0
\]

As for the transversal waves we try the plane wave ansatz and assume \( \text{k} \) to be real.

\[
\vec{E}(r, \omega) = \vec{E}(\omega) \exp(\text{i}kr)
\]

With \( \text{rot} \left[ \vec{E}(\omega) \exp(\text{i}kr) \right] = \text{i} \text{k} \times \vec{E}(\omega) \exp(\text{i}kr) \) we get from the wave equation:

\[
\text{k} \times \left[ \text{k} \times \vec{E}(r, \omega_L) \right] = 0
\]

Now we decompose the electric field into transversal and longitudinal components with respect to the wave vector:

\[
\vec{E}(r, \omega) = \vec{E}(\omega) \exp(\text{i}kr) = \vec{E}_{\perp}(\omega) \exp(\text{i}kr) + \vec{E}_z(\omega) \exp(\text{i}kr)
\]

with \( \vec{E}_{\perp}(\omega) \perp \text{k} \) and \( \vec{E}_z(\omega) \parallel \text{k} \)

This decomposed field is inserted into the wave equation:

\[
\text{k} \times \left[ \text{k} \times (\vec{E}_z + \vec{E}_{\perp}) \right] \exp(\text{i}kr) = 0
\]

\[
\text{k} \times \left[ \text{k} \times \vec{E}_z \right] \exp(\text{i}kr) + \text{k} \times \left[ \text{k} \times \vec{E}_{\perp} \right] \exp(\text{i}kr) = 0
\]

Since the cross product of \( \text{k} \) with the longitudinal field \( \vec{E}_z(\omega) \) is trivially zero the remaining wave equation is:

\[
k^2 \vec{E}_z = 0
\]

Hence the transversal field \( \vec{E}_{\perp} \) must vanish and the only remaining field component is the longitudinal field \( \vec{E}_z(\omega) \):

\[
\rightarrow \vec{E}(r, \omega_L) = \vec{E}_z(\omega) \exp(\text{i}kr)
\]

**2.4.3 Plane wave solutions in different frequency regimes**

The dispersion relation for plane wave solutions \( k^2 = k^2 + k_{\perp}^2 + k_{\parallel}^2 = \varepsilon(\omega) \) dictates the (complex) wavenumber \( k \) only. Thus, different solutions for the complex wave vector \( \text{k} = \text{k}' + \text{i}k'' \) are possible. In addition, the generalized dielectric function \( \varepsilon(\omega) \) is complex. In this chapter we will discuss possible scenarios and resulting plane wave solutions.

**A) Positive real valued epsilon** \( \varepsilon(\omega) = \varepsilon'(\omega) > 0 \)

This is the regime favorable for optics. We have transparency, and the frequency is far from resonances. The dispersion relation gives

\[
k^2 = k^2 - k_{\perp}^2 + 2\text{i}k' \cdot k'' = \frac{\omega^2}{c^2} \varepsilon'(\omega) = \frac{\omega^2}{c^2} n'(\omega) \quad \Rightarrow \quad \text{k}' \cdot \text{k}'' = 0
\]

There are two possibilities to fulfill this condition, either \( k'' = 0 \) or \( k' \perp k'' \).

**A.1) Real valued wave-vector** \( k'' = 0 \)

- In this case the wave vector is real and we find the dispersion relation

\[
k'(\omega) = \frac{\omega}{c} n(\omega) = \frac{\omega}{c} \frac{2\pi}{\lambda} n(\omega)
\]

- Because \( k'' = 0 \) these waves are homogeneous, i.e. planes of constant phase are parallel to the planes of constant amplitude. This is trivial, because the amplitude is constant.

**Example 1:** single resonance in dielectric material

- for lattice vibrations (phonons)
Now the imaginary part of \( \varepsilon(\omega) \) is neglected, which mathematically corresponds to an undamped resonance

\[
\varepsilon(\omega) = \varepsilon'(\omega) = \varepsilon_\infty + \frac{\omega_0^2}{\omega^2 - \omega_0^2}
\]

- We can invert the dispersion relation \( \omega(k) = \frac{\omega}{c} \sqrt{\varepsilon(\omega)} \to \omega(k) \):

\[
\omega = \frac{\omega}{c} \sqrt{\varepsilon(\omega)}
\]

Example 2: free electrons
- for plasma and metal
- Again the imaginary part of \( \varepsilon(\omega) \) is neglected

\[
\varepsilon(\omega) = \varepsilon'(\omega) = 1 - \frac{\omega_p^2}{\omega^2}
\]

- We again invert the dispersion relation \( k(\omega) = \frac{\omega}{c} \sqrt{\varepsilon(\omega)} \to \omega(k) \):


A.2) Complex valued wave-vector \( k' \perp k'' \)

- The second possibility to fulfill the dispersion relation leads to a complex wave-vector and so-called evanescent waves. We find

\[
k^2 = k'^2 - k''^2 = \frac{\omega^2}{c^2} \varepsilon(\omega)
\]

and therefore \( k''^2 = k^2 - k'^2 \)

- This means that

\[
k''^2 \neq 0 \quad \text{and} \quad k^2 > k'^2
\]

- We will discuss the importance of evanescent waves in the next chapter, where we will study the propagation of arbitrary initial field distributions. What is interesting to note here is that evanescent waves can have arbitrary large \( k^2 > k'^2 \), whereas the homogeneous waves of case A.1) ( \( k'' = 0 \) ) obey \( k'^2 = k^2 \). If we plug our findings into the plane wave ansatz we get: for the evanescent waves:

\[
E(r, \omega) = E(\omega) \exp \left( \frac{1}{i} \left[ k'(\omega)r \right] \right) \exp(-k''(\omega)r)
\]

- The planes defined by the equation \( k''(\omega)r = \text{const.} \) are the so-called planes of constant amplitude, those defined by \( k'(\omega)r = \text{const.} \) are the planes of constant phase. Because of \( k' \perp k'' \) these planes are perpendicular to each other.

- The factor \( \exp(-k''(\omega)r) \) leads to exponential growth of evanescent waves in homogeneous space. Therefore, evanescent waves can't be physically justified normal modes of homogeneous space and can only exist in inhomogeneous space, where the exponential growth is suppressed, e.g. at interfaces.

B) Negative real valued epsilon \( \varepsilon(\omega) = \varepsilon'(\omega) < 0 \)

This situation (negative but real \( \varepsilon(\omega) \) can occur near resonances in dielectrics ( \( \omega_0 < \omega < \omega_c \)) or below the plasma frequency ( \( \omega < \omega_p \)) in metals. Then the dispersion relation gives
As in the previous case A), the imaginary term has to vanish and $k' k'' = 0$. Again this can be achieved by two possibilities.

**B.1)** $k' = 0$

$\Rightarrow k'' = \frac{\varepsilon''}{c^2} \epsilon'(\omega) \Rightarrow E(r, \omega) \sim \exp(-k'' r) \Rightarrow$ strong damping

**B.2)** $k' \cdot k'' = 0$

$\Rightarrow k' \perp k'' \Rightarrow$ evanescent waves

$$k^2 = k'^2 - k''^2 = -\frac{\varepsilon''}{c^2} \epsilon'(\omega)$$

$$k''^2 = \frac{\varepsilon''}{c^2} \epsilon'(\omega) + k'^2.$$  

As above, these evanescent waves exist only at interfaces (like for $\varepsilon'(\omega) = \epsilon'(\omega) > 0$). The interesting point is that here we find evanescent waves for all values of $k'$. In particular, case B.1) ($k' = 0$) is included. Hence, we can conclude that for $\varepsilon'(\omega) = \epsilon'(\omega) < 0$ we find only evanescent waves!

**C) Complex valued epsilon $\varepsilon(\omega)$**

This is the general case, which is relevant particularly near resonances. From our (optical) point of view only weak absorption is interesting. Therefore, in the following we will always assume $\varepsilon''(\omega) << \epsilon'(\omega)$. As we can see in the following sketch, we can have $\epsilon'(\omega) > 0, \epsilon''(\omega) > 0$, or $\epsilon'(\omega) < 0, \epsilon''(\omega) > 0$.

Let us further consider only the important special case of quasi-homogeneous plane waves, i.e., $k'$ and $k''$ are almost parallel. Then, it is convenient to use the complex refractive index

$$[k' + ik'']^2 = k^2(\omega) = \frac{\varepsilon''}{c^2} \epsilon'(\omega) = \frac{\varepsilon''}{c^2} n^2(\omega) = \frac{\varepsilon''}{c^2} [n(\omega) + i \kappa(\omega)]^2.$$  

Since $k'$ and $k''$ are almost parallel:

$$\Rightarrow |k'| = \frac{\varepsilon''}{c^2} n(\omega), \quad |k''| = \frac{\varepsilon''}{c^2} \kappa(\omega).$$

The dispersion relation in terms of the complex refractive index gives

$$k^2 = k'^2 = \frac{\varepsilon''}{c^2} \epsilon'(\omega) = \frac{\varepsilon''}{c^2} [n(\omega) + i \kappa(\omega)]^2.$$  

Here we have

$$\epsilon'(\omega) = \epsilon'(\omega) + \frac{\varepsilon''(\omega)}{\varepsilon'(\omega)} = n^2(\omega) - \kappa(\omega) + 2i n(\omega) \kappa(\omega),$$

and therefore $\epsilon'(\omega) = n^2(\omega) - \kappa^2(\omega)$

$$\epsilon''(\omega) = 2n(\omega) \kappa(\omega)$$

$$n^2(\omega) = \frac{\varepsilon''}{2} \left[ \text{sgn} \left( \epsilon' \right) \sqrt{1 + \left( \epsilon'/\epsilon'' \right)^2} + 1 \right].$$

$$\kappa^2(\omega) = \frac{\varepsilon''}{2} \left[ \text{sgn} \left( \epsilon' \right) \sqrt{1 + \left( \epsilon'/\epsilon'' \right)^2} - 1 \right].$$

Two important limiting cases of quasi-homogeneous plane waves:

**C.1)** $\epsilon', \epsilon'' > 0$, $\epsilon'' << \epsilon'$, **(dielectric media)**

$$n(\omega) \approx \sqrt{\epsilon'(\omega)}, \quad \kappa(\omega) \approx \frac{1}{2} \frac{\epsilon'(\omega)}{\sqrt{\epsilon'(\omega)}}$$

$$\Rightarrow \epsilon'(\omega) = \epsilon'(\omega) + \frac{\epsilon''(\omega)}{\epsilon'(\omega)} = n^2(\omega) - \kappa^2(\omega) + 2i n(\omega) \kappa(\omega).$$

and therefore $\epsilon'(\omega) = n^2(\omega) - \kappa^2(\omega)$

$$\epsilon''(\omega) = 2n(\omega) \kappa(\omega)$$

$$n^2(\omega) = \frac{\epsilon''}{2} \left[ \text{sgn} \left( \epsilon' \right) \sqrt{1 + \left( \epsilon'/\epsilon'' \right)^2} + 1 \right].$$

$$\kappa^2(\omega) = \frac{\epsilon''}{2} \left[ \text{sgn} \left( \epsilon' \right) \sqrt{1 + \left( \epsilon'/\epsilon'' \right)^2} - 1 \right].$$

Two important limiting cases of quasi-homogeneous plane waves:
In this regime propagation dominates \((n(\omega) \gg \kappa(\omega))\), and we have weak absorption:

\[
\begin{align*}
k'^2 - k''^2 &= \frac{\omega^2}{c^2} \varepsilon'(\omega), \\
2k' \cdot k'' &= \frac{\omega^2}{c^2} \varepsilon''(\omega), \\
|k'| &= \frac{\omega}{c} n(\omega) \approx \frac{\omega}{c} \sqrt{\varepsilon'(\omega)}, \\
|k''| &= \frac{\omega}{c} \kappa(\omega) \approx \frac{1}{2} \frac{\omega}{c} \frac{\varepsilon'(\omega)}{\sqrt{\varepsilon'(\omega)}} \\
- k' \cdot k'' &\approx |k'||k''|.
\end{align*}
\]

- \(k'\) and \(k''\) almost parallel \(\rightarrow\) homogeneous waves

\(\rightarrow\) in homogeneous, isotropic media, next to resonances, we find damped, homogeneous plane waves, \(k' || k'' || e_k\) with \(e_k\) being the unit vector along \(k\)

\[
\mathbf{E}(r, \omega) = E(0) \exp(i k r) = E(0) \operatorname{exp} \left[ i \left( \frac{\omega}{c} n(\omega) (e_k \cdot r) \right) \right] \operatorname{exp} \left[ -i \frac{\omega}{c} \kappa(\omega) (e_k \cdot r) \right].
\]

\[C.2)\] \(\varepsilon' < 0, \varepsilon'' > 0, \varepsilon'' \ll |\varepsilon'|\), (metals and dielectric media in so-called Reststrahl domain)

\[
n(\omega) \approx \frac{1}{2} \frac{\varepsilon'(\omega)}{\sqrt{\varepsilon'(\omega)}}, \quad \kappa(\omega) \approx \frac{\varepsilon'(\omega)}{\sqrt{\varepsilon'(\omega)}}.
\]

In this regime damping dominates \((n(\omega) \ll \kappa(\omega))\) and we find a very small refractive index. Interestingly, propagation (nonzero \(n\)) is only possible due to absorption (see time averaged Poynting vector below).

**Summary of normal modes**

- undamped homogeneous waves and evanescent waves
- evanescent waves
- weakly damped quasi-homogeneous waves
- strongly damped quasi-homogeneous waves

### 2.4.4 Time averaged Poynting vector of plane waves

\[
\langle \mathbf{S}(r, t) \rangle = \frac{1}{2} \frac{\mathbf{E}^\ast(t) \times \mathbf{H}'(r,t)}{\mu_0} dt,
\]

For plane waves we find:

\[
E(r, t) = E(0) \exp(i \mathbf{k} \cdot \mathbf{r} - i \omega t) = E(0) \exp(i \mathbf{k}' \cdot \mathbf{r} - i \omega t)
\]

\[
H(r, t) = \frac{1}{\omega \mu_0} \mathbf{k} \times E(r, t)
\]

assuming a stationary case \(E(t) = E(0) \exp(-i \omega t)\)

\[
\rightarrow \mathbf{S}(r, t) = \frac{1}{2} \frac{\mathbf{k}'}{2 \omega \mu_0} \exp[-2 \mathbf{k}' \cdot \mathbf{r}] E(0)^2 \mathbf{E}^\ast \mathbf{E} = \frac{1}{2} \frac{\mathbf{E}^\ast}{\mu_0} \mathbf{E} \mathbf{E}^\ast - 2 \frac{\omega}{c} \kappa(\omega) (e_k \cdot \mathbf{r}) |\mathbf{E}|^2
\]

with \(e_k\) being the unit vector along \(k'\) and \(e_{k'}\) being the unit vector along \(k^\ast\).

### 2.5 The Kramers-Kronig relation

In the previous sections we have assumed a very simple model for the description of the material’s response to the excitation by the electromagnetic field. This model was based on quite strong assumptions, like a single charge which is attached to a rigid lattice etc. Hence, one could imagine that more complex matter could give rise to arbitrarily complex response functions if adequate models would be used for its description. However we can show from basic laws of physics, that several properties are common to all possible response functions, as long as a linear response to the excitation is assumed.

These fundamental properties of the response function are formulated mathematically by the Kramers-Kronig relation. It is a general relation between \(\varepsilon'(\omega)\) (dispersion) and \(\varepsilon''(\omega)\) (absorption). This means in practice that we can compute \(\varepsilon'(\omega)\) from \(\varepsilon''(\omega)\) and vice versa. For example, if we have access to the absorption spectrum of a medium, we can calculate the dispersion.

The Kramers-Kronig relation follows from reality and causality of the response function \(R\) of a linear system. That the response function is real valued is a direct consequence from Maxwell’s equations which are real valued as well. Causality is also a very fundamental property, since the polarization must not depend on some future electric field. As we have seen in the previous sections, in time-domain the polarization and the electric field are related as:

\[
P_i(t, \tau) = e_0 \int_{-\infty}^{t} R(t - \tau') E_i(\tau') d\tau' \leftrightarrow P_e(t, \tau) = e_0 \int_{-\infty}^{t} R(\tau) E_e(r, t - \tau) d\tau
\]

Reality of the response function implies:
Causality of the response function implies:

\[ R(\tau) = 0(\tau) y(\tau) \quad \text{with} \quad 0(\tau) = \begin{cases} 
1 & \text{for } \tau > 0 \\
\frac{1}{2} & \text{for } \tau = 0 \\
0 & \text{for } \tau < 0 
\end{cases} \rightarrow \text{Heaviside distribution} \]

In the following, we will make use of the Fourier transform of Heaviside distribution:

\[ 2\pi \delta(o) = \int_{-\infty}^{\infty} dt \delta(t) \rightarrow \text{defined as integral only} \]

In Fourier space, the Heaviside distribution consists of the Dirac delta distribution:

\[ \int_{-\infty}^{\infty} d\omega \delta(\omega - \omega_0) f(\omega) = f(\omega_0) \rightarrow \text{Dirac delta distribution} \]

and the expression \( P(\delta) \) involving a Cauchy principal value:

\[ P \int_{-\infty}^{\infty} \frac{d\omega}{\omega} f(\omega) = \lim_{\omega \to 0} \left[ \int_{-\infty}^{0} \frac{d\omega}{\omega} f(\omega) + \int_{0}^{\infty} \frac{d\omega}{\omega} f(\omega) \right] \rightarrow \text{Cauchy principal value} \]

As we have seen above, causality implies that the response function has to contain a multiplicative Heaviside function. Hence, in Fourier space (susceptibility) we expect a convolution:

\[ \chi(\omega) = \int_{-\infty}^{\infty} d\tau R(\tau) e^{i\omega \tau} = \int_{-\infty}^{\infty} d\tau 0(\tau) y(\tau) e^{i\omega \tau} \]

\[ = \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} P \frac{d}{\omega} \frac{i}{\omega} \frac{\omega(\omega)}{\omega - \omega} \]

In order to derive the Kramers-Kronig relation we can use a small trick (this trick saves us using complex integration in the derivation). Because of the Heaviside function, we can choose the function \( y(\tau) \) for \( \tau < 0 \) arbitrarily without altering the susceptibility! In particular, we can choose:

\[ a) \quad y(-\tau) = y(\tau) \quad \text{even function} \]
\[ b) \quad y(-\tau) = -y(\tau) \quad \text{odd function} \]

a) \( y(-\tau) = y(\tau) \)

In this case \( y(-\tau) = y(\tau) \) is a real valued and even function. We can exploit this property and show that

\[ \phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau y(\tau) e^{i\omega \tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau y(\tau) e^{-i\omega \tau} = \phi^*(\omega) \] is real as well

Hence, we can conclude from equation (*) above that

\[ \chi^*(\omega) = -\frac{1}{2\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{i}{\omega} \frac{\omega(\omega)}{\omega - \omega} \frac{\omega(\omega)}{\omega - \omega} \]

Here \( P \int_{-\infty}^{\infty} \) is a so called principal value integral (G: Hauptwertintegral).

Now we have expressions for \( \chi(\omega), \chi^*(\omega) \) and can compute real and imaginary part of the susceptibility:

\[ \chi(\omega) + \chi^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{i}{\omega} \frac{\omega(\omega)}{\omega - \omega} + \frac{\omega(\omega)}{\omega - \omega} - \frac{1}{2\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \omega} \frac{\omega(\omega)}{\omega - \omega} \frac{\omega(\omega)}{\omega - \omega} = \phi(\omega) \]

\[ \chi(\omega) - \chi^*(\omega) = \ldots = -\frac{1}{2\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \omega} \frac{\omega(\omega)}{\omega - \omega} \]

Plugging the last two equations together we find the first Kramers-Kronig relation:

\[ \Re \chi(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \omega} \frac{\omega(\omega)}{\omega - \omega} \]

1. K-K relation

Knowledge of the real part of the susceptibility (dispersion) allows us to compute the imaginary part (absorption).

b) \( y(-\tau) = -y(\tau) \)

The second K-K relation can be found by a similar procedure when we assume that \( y(-\tau) = -y(\tau) \) is a real odd function. We can show that in this case

\[ \phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau y(\tau) e^{i\omega \tau} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau y(\tau) e^{-i\omega \tau} = \phi^*(\omega) \] is purely imaginary

With equation (*) we then find that

\[ \chi^*(\omega) = -\frac{1}{2\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \omega} \frac{\omega(\omega)}{\omega - \omega} \]

(see (*) ) and

Again we can then compute real and imaginary part of the susceptibility
\[
\chi(\omega) - \chi^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i \mathcal{Y}(\omega)}{\omega - \omega_0} d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i \overline{\mathcal{Y}(\omega)}}{\omega - \omega_0} d\omega = \mathcal{Y}(\omega)
\]

\[
\chi(\omega) + \chi^*(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{Y}(\omega)}{\omega - \omega_0} d\omega
\]

and finally obtain

\[
\Re \chi(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{Y}(\omega)}{\omega - \omega_0} d\omega
\]

2. K-K relation

The second Kramers-Kronig relation allows us to compute the real part of the susceptibility (dispersion) when we know its imaginary part (absorption).

The Kramers-Kronig relation can also be rewritten in terms of the dielectric function, where one applies also the symmetry relation for \( \omega \):

K-K relation for \( \varepsilon \):

\[
\chi(\omega) = \chi^*(\omega) \rightarrow \chi'(\omega) = \chi^*(\omega) \quad \chi^*(\omega) = -\chi'(\omega)
\]

\[
\chi(\omega) = \varepsilon(\omega) - 1 = [\varepsilon'(\omega) - 1] + i \chi'(\omega)
\]

\[
\varepsilon'(\omega) - 1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon'(\omega)}{\omega - \omega_0} d\omega,
\]

\[
\varepsilon'(\omega) = \frac{2}{\pi\omega} \int_{-\infty}^{\infty} \frac{[\varepsilon'(\omega) - 1]}{\omega - \omega_0^2} d\omega.
\]

- dispersion and absorption are linked, e.g., we can measure absorption and compute dispersion

Example:

\[
\varepsilon'(\omega) - \delta(\omega - \omega_0) \rightarrow \varepsilon'(\omega) - 1 \sim \frac{\omega_0}{\omega_0 - \omega_0} \rightarrow \text{Drude-Lorentz model}
\]

### 2.6 Beams and pulses - analogy of diffraction and dispersion

In this chapter we will analyze the propagation of light. In particular, we will answer the question how an arbitrary beam (spatial) or pulse (temporal) will change during propagation in isotropic, homogeneous, dispersive media. Relevant (linear) physical effects are diffraction and dispersion. Both phenomena can be understood very easily in the Fourier domain. Temporal effects, i.e. the dispersion of pulses, will be treated in temporal Fourier domain (temporal frequency domain). Spatial effects, i.e. the diffraction of beams, will be treated in the spatial Fourier domain (spatial frequency domain). We will see that:

- Pulses with finite spatial width (i.e. pulsed beams) are superposition of normal modes (in frequency- and spatial frequency domain).
- Spatio-temporally localized optical excitations delocalize during propagation because of different phase evolution for different frequencies and spatial frequencies (different propagation directions of normal modes).

Let us have a look at the different possibilities (beam, pulse, pulsed beam)

**A) beam → finite transverse width → diffraction**

A beam is a continuous superposition of stationary plane waves (normal modes) with different wave vectors (propagation directions).

\[
E(r, t) = \int_{-\infty}^{\infty} E(k) \exp \left[ i \left( k r - \omega t \right) \right] d^3k
\]

**B) pulse → finite duration → dispersion**

A pulse is a continuous superposition of stationary plane waves (normal modes) with different frequencies.
2.7 Diffraction of monochromatic beams in homogeneous isotropic media

Let us have a look at the propagation of monochromatic beams first. In this situation, we have to deal with diffraction only. We will see later that pulses and their dispersion can be treated in a very similar way. Treating diffraction in the framework of wave-optical theory (or even Maxwell) allows us to treat rigorously many important optical systems and effects, i.e., optical imaging and resolution, filtering, microscopy, gratings, ...

In this chapter, we assume stationary fields and therefore $\omega = \text{const.}$ For technical convenience and because it is sufficient for many important problems, we will make the following assumptions and approximations:

- $\varepsilon(\omega) = \varepsilon'(\omega) > 0$, \( \rightarrow \) optical transparent regime \( \rightarrow \) normal modes are stationary homogeneous and evanescent plane waves
- scalar approximation

\[
E(r, t) = \int_{-\infty}^{\infty} E(k, \omega) \exp\left[i(k \cdot r - \omega t)\right] d\omega
\]

C) pulsed beams \( \rightarrow \) finite transverse width and finite duration \( \rightarrow \) diffraction and dispersion

A pulsed beam is a continuous superposition of stationary plane waves (normal modes) with different frequency and different propagation direction

\[
E(r, t) = \int_{-\infty}^{\infty} E(k, \omega) \exp\left[i(k \cdot r - \omega t)\right] dk d\omega
\]

2.7.1 Arbitrarily narrow beams (general case)

Let us consider the following fundamental problem. We want to compute from a given field distribution $u(x, y, 0)$ in the plane $z = 0$ the complete field $u(x, y, z)$ in the half-space $z > 0$, where $z$ is our "propagation direction".

The governing equation is the scalar Helmholtz equation

\[
\Delta u(r, \omega) + \frac{\omega^2}{c^2} \varepsilon(\omega) u(r, \omega) = 0
\]

In the last step we inserted the dispersion relation (wave number $k(\omega)$). In the following we often even omit the argument of the fixed frequency $\omega$.

To solve this equation and to calculate the dynamics of the fields, we can switch again to the Fourier domain.

We take the Fourier transform

\[
\mathcal{F}\{u(x, y, 0)\} = U(k, \omega)
\]

which can be interpreted as a superposition of normal modes with different propagation directions and wavenumbers $k(\omega)$ (here the absolute value of the wave-vector $k$). Naively, we could expect that we just constructed a general solution to our problem, but the solution is not correct because of the dispersion relation:

\[
k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \varepsilon(\omega)
\]

\( \rightarrow \) only two components of $k$ are independent, e.g., $k_x, k_y$.

Our naming convention is in the following: $k_x = \alpha, \quad k_y = \beta, \quad k_z = \gamma$. 

Then, the dispersion relation reads:
\[ k^2(\omega) = \alpha^2 + \beta^2 + \gamma^2 \]

Thus, to solve our problem we need only a two-dimensional Fourier transform, with respect to transverse directions to the “propagation direction \( z \)”: 
\[
\mathcal{U}(r) = \iiint_{-\infty}^{\infty} U(\alpha, \beta, z) \exp\left[i(\alpha x + \beta y)\right]d\alpha d\beta.
\]

In analogy to the frequency \( \omega \) we call \( \alpha, \beta \) spatial frequencies.

Now we plug this expression into the scalar Helmholtz equation 
\[
\Delta \mathcal{U}(r) + k^2(\omega)\mathcal{U}(r) = 0
\]

This way we can transfer the Helmholtz equation in two spatial dimensions into Fourier space
\[
\left( \frac{d^2}{dz^2} + k^2 - \alpha^2 - \beta^2 \right) U(\alpha, \beta, z) = 0,
\]
\[
\left( \frac{d^2}{dz^2} + \gamma^2 \right) U(\alpha, \beta, z) = 0.
\]

This equation is easily solved and yields the general solution
\[
U(\alpha, \beta, z) = U_1(\alpha, \beta) \exp[i\gamma(\alpha, \beta)z] + U_2(\alpha, \beta) \exp[-i\gamma(\alpha, \beta)z],
\]
depending on \( \gamma(\alpha, \beta) = \sqrt{k^2(\omega) - \alpha^2 - \beta^2} \).

We can identify two types of solutions:

**A) Homogeneous waves**
\( \gamma \geq 0, \rightarrow \alpha^2 + \beta^2 \leq k^2 \), i.e., \( k \) real \( \rightarrow \) homogeneous waves

**B) Evanescent waves**
\( \gamma < 0, \rightarrow \alpha^2 + \beta^2 > k^2 \), i.e., \( k \) complex, because \( \gamma = k_z \) imaginary. Then, we have \( k = k' + i k'' \), with \( k' = \alpha e_x + \beta e_y \) and \( k'' = \gamma e_z \).
\( \rightarrow k' \perp k'' \rightarrow \) evanescent waves

We see immediately that in the half-space \( z > 0 \) the solution \( \sim \exp(-\gamma z) \) grows exponentially. Because this does not make sense, this component of the solution must vanish \( U_2(\alpha, \beta) = 0 \). In fact, we will see later that \( U_2(\alpha, \beta) \) corresponds to backward running waves, i.e., light propagating in the opposite direction. We therefore find the solution:
\[
U(\alpha, \beta, z) = U_1(\alpha, \beta) \exp[i\gamma(\alpha, \beta)z],
\]
\[
= U(\alpha, \beta, 0) \exp[i\gamma(\alpha, \beta)z] + \hat{U}_0(\alpha, \beta) \exp[i\gamma(\alpha, \beta)z] \]

Furthermore the following boundary condition holds: \( U(\alpha, \beta, 0) = U_0(\alpha, \beta) \).

In spatial space, we can find the optical field for \( z > 0 \) by inverse Fourier transform:
\[
\mathcal{U}(r) = \iiint_{-\infty}^{\infty} U(\alpha, \beta, z) \exp[i(\alpha x + \beta y)]d\alpha d\beta.
\]

For homogeneous waves (real \( \gamma \)) the red term above causes a certain phase shift for the respective plane wave during propagation. Hence, we can formulate the following result:

**Diffraction** is due to different phase shifts in propagation direction for the different normal modes according to their different spatial frequencies \( \alpha, \beta \).

The initial spatial frequency spectrum or angular spectrum at \( z = 0 \) forms the initial condition of the initial value problem and follows from \( u_0(x, y) = u(x, y, 0) \) by Fourier transform:
\[
U_0(\alpha, \beta) = \left( \frac{1}{2\pi} \right)^2 \iiint_{-\infty}^{\infty} u_0(x, y) \exp[-i(\alpha x + \beta y)]dxdy.
\]

As mentioned above the wave-vector components \( \alpha, \beta \) are the so-called spatial frequencies. Another common terminology is “direction cosine” for the quantities \( \alpha/k, \beta/k \), because of the direct link to the angle of the respective
plane wave. For example \( \alpha / k = \cos \theta \) gives the angle of the plane wave’s propagation direction with the \( x \)-axis.

**Scheme for calculation of beam diffraction**

We can formulate a general scheme to describe the diffraction of beams:

1. initial field: \( u_0(x, y) \)
2. initial spectrum: \( U_0(\alpha, \beta) \) by Fourier transform
3. propagation: by multiplication with \( \exp[i \gamma(\alpha, \beta)z] \)
4. new spectrum: \( U(\alpha, \beta, z) = U_0(\alpha, \beta) \exp[i \gamma(\alpha, \beta)z] \)
5. new field distribution: \( u(x, y, z) \) by Fourier back transform

This scheme allows for two interpretations:

1) The resulting field distribution is the Fourier transform of the propagated spectrum

\[
u(r) = \int_{-\infty}^{\infty} U(\alpha, \beta, z) \exp[i (\alpha x + \beta y)] d\alpha d\beta.
\]

2) The resulting field distribution is a superposition of homogeneous and evanescent plane waves (‘plane-wave spectrum’) which obey the dispersion relation

\[
u(r) = \int_{-\infty}^{\infty} U_h(\alpha, \beta) \exp[i (\alpha x + \beta y + \gamma(\alpha, \beta)z)] d\alpha d\beta.
\]

Let us now discuss the complex transfer function \( H(\alpha, \beta; z) = \exp[i \gamma(\alpha, \beta)z] \), which describes the beam propagation in Fourier space. For \( z = \text{const.} \) (finite propagation distance) it looks like:

\[
\begin{align*}
\text{amplitude} & : H(\alpha, \beta; z) = \exp[i \gamma(\alpha, \beta)z] \quad \text{for } \alpha^2 + \beta^2 \leq k^2 \\
\text{phase} & : \arg[H(\alpha, \beta; z)] = \arg[\exp[i \gamma(\alpha, \beta)z]] = \begin{cases} 1 & \text{for } \alpha^2 + \beta^2 \leq k^2 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Obviously, \( H(\alpha, \beta; z) = \exp[i \gamma(\alpha, \beta)z] \) acts differently on homogeneous and evanescent waves:

**A) homogeneous waves** \( \rightarrow \alpha^2 + \beta^2 \leq k^2 \)

\[
\exp[i \gamma(\alpha, \beta)z] = 1, \quad \arg[\exp[i \gamma(\alpha, \beta)z]] = 0
\]

Upon propagation the homogeneous waves are multiplied by the phase factor

\[
\exp[i \sqrt{k^2 - \alpha^2 - \beta^2} z]
\]

**B) evanescent waves** \( \rightarrow \alpha^2 + \beta^2 > k^2 \)

\[
\exp[i \sqrt{\alpha^2 + \beta^2 - k^2} z], \quad \arg[\exp[i \gamma(\alpha, \beta)z]] = 0
\]

Upon propagation the evanescent waves are multiplied by an amplitude factor <1

\[
\exp[-\sqrt{\alpha^2 + \beta^2 - k^2} z] < 1
\]

This means that their contribution gets damped with increasing propagation distance \( z \).

Now the question is: When do we get evanescent waves? Obviously, the answer lies in the boundary condition: Whenever \( u_0(x, y) \) yields an angular spectrum \( U_0(\alpha, \beta) \neq 0 \) for \( \alpha^2 + \beta^2 > k^2 \) we get evanescent waves.

**Example: Slit**

Let us consider the following one-dimensional initial condition which corresponds to an aperture of a slit:

\[
u_0(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{a}{2} \\
0 & \text{otherwise}
\end{cases}
\]

\[
U_0(\alpha) = \text{FT}[u_0(x)] \sim \frac{\sin \left( \frac{a \alpha}{2} \right)}{\left( \frac{a \alpha}{2} \right)} = \text{sinc} \left( \frac{a \alpha}{2} \right)
\]
- All spatial frequencies \((-\infty \rightarrow \infty)\) are excited.
- Important spectral information is contained in the interval \(\frac{2}{\alpha}\).
- Largest important spectral frequency for a structure with width \(a\) is \(\frac{2}{\alpha} = \pi\).
- Evanescent waves appear for \(k > \alpha\).
- To represent the relevant information by homogeneous waves the following condition must be fulfilled: \(\frac{2}{a} < k = \frac{2}{n} \rightarrow a > \frac{\lambda}{n}\).

**General result**

We have seen in the example above that evanescent waves appear for structures \(<\) wavelength in the initial condition. Information about those small structures gets lost for \(z \gg \lambda\).

**Conclusion**

In homogeneous media, only information about structural details having length scales of \(|\Delta \lambda|, |\Delta \nu| > \lambda / n\) are transmitted over macroscopic distances. Homogeneous media act like a low-pass filter for light.

**Summary of beam propagation scheme**

\[ u_0(x,y) \xrightarrow{\text{FT}} u_0(\alpha,\beta) \quad \rightarrow \quad U(\alpha,\beta;z) = H(\alpha,\beta;z)U_0(\alpha,\beta) \xrightarrow{\text{FT}} u(x,y,z) \]

with the transfer function \(H(\alpha,\beta;z) = \exp[iC_2(\alpha,\beta)z]\)

**Remark: diffraction free beams**

With our understanding of diffraction it is straightforward to construct so-called diffraction free beams, i.e., beams that do not change their amplitude distribution during propagation. Translated to Fourier space this means that all spatial frequency components have to get the same phase shift during the propagation

\[ U(\alpha,\beta;z) = U_0(\alpha,\beta)\exp[iC_2(\alpha,\beta)z] = U_0(\alpha,\beta)\exp[iC_2] \]

\[ u(x,y,z) = \exp[iC_2]u_0(x,y) \]

Since in general \(\gamma(\alpha,\beta) \neq \text{const}\) the excitation \(u_0(x,y)\) must have a shape such that its Fourier transform has only components where the transfer function is of equivalent value

\[ U_0(\alpha,\beta) \neq 0 \text{ only for } \gamma(\alpha,\beta) = \sqrt{k^2 - \alpha^2 - \beta^2} = C \]

It is straightforward to see that the excited spatial frequencies must lie on a circular ring in the \((\alpha,\beta)\) plane.

\[ \alpha^2 + \beta^2 = \rho_0^2 \]

For constant spectral amplitude on this ring the Fourier back-transform yields (see exercises):

\[ u_0(x,y) = J_0(pr) \]

**2.7.2 Fresnel- (paraxial) approximation**

The beam propagation formalism developed in the previous chapter can be simplified for the important special case of a narrowband angular spectrum

\[ U_0(\alpha,\beta) \neq 0 \text{ for } \alpha^2 + \beta^2 \ll k^2 \]

In this situation the beam consists of plane waves having only small inclination with respect to the optical \(z\)-axis (paraxial (Fresnel) approximation). Then, we can simplify the expression for \(\gamma(\alpha,\beta)\) by a Taylor expansion to:

\[ \gamma(\alpha,\beta) = \sqrt{k^2 - \alpha^2 - \beta^2} \approx k \left(1 - \frac{\alpha^2 + \beta^2}{2k^2}\right) = k - \frac{\alpha^2 + \beta^2}{2k} \]

The resulting expression for the transfer function in Fresnel approximation reads:
We can see that this $H_f(\alpha, \beta; z)$ is always real valued. Hence it does not account for the physics of evanescent waves. However, if we remember that for the derivation of the $H_f(\alpha, \beta; z)$ as an approximation of $H(\alpha, \beta; z)$ we had assumed that the spatial frequency spectrum is narrow (paraxial waves) which had excluded the excitation of evanescent waves already from the beginning to justify the paraxial approximation.

The assumption of a narrow frequency spectrum corresponds to the requirement that all structural details $|\Delta x|, |\Delta y|$ of the field distribution in the excitation plane ($z = 0$) must be much larger than the wavelength:

$$|\Delta x|, |\Delta y| > 10 \lambda, n \gg \lambda, n$$

This requirement applies also to the phase of the excitation. Hence it is not sufficient that only the structural details of the intensity have a large scale. The underlying phase of the excitation field must fulfill this condition as well.

The propagation of the spectrum in Fresnel approximation works in complete analogy to the general case. We just use the modified transfer function to describe the propagation:

$$U_f(\alpha, \beta; z) = H_f(\alpha, \beta; z)U_0(\alpha, \beta)$$

**Summary of Fresnel approximation**

For a coarse initial field distribution $u_0(x, y, z)$ the angular spectrum $U_0(\alpha, \beta)$ is nonzero for $\alpha^2 + \beta^2 \ll k^2$ only. Then, only paraxial plane waves are relevant for transmitting information and the transfer function of homogeneous space can be approximated by $H_f(\alpha, \beta; z)$.

**Description in real space**

It is also possible to formulate beam propagation in Fresnel (paraxial) approximation in position space:

$$u_f(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_f(\alpha, \beta; z) \exp[i(\alpha x + \beta y)] d\alpha d\beta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_f(\alpha, \beta; z)U_0(\alpha, \beta) \exp[i(\alpha x + \beta y)] d\alpha d\beta$$

$$= \int_{-\infty}^{\infty} h_f(x-x', y-y'; z)u_0(x', y') dx'dy'$$

The spatial response function $h_f(x, y; z)$ follows from the convolution theorem and is the Fourier transform of $H_f(\alpha, \beta; z)$:

$$h_f(x, y; z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} H_f(\alpha, \beta; z) \exp[i(\alpha x + \beta y)] d\alpha d\beta$$

$$= \left(\frac{1}{2\pi}\right)^2 \exp[i(kz)] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2k} \left(\alpha^2 + \beta^2\right) - \frac{(x-x')^2}{2z^2} - \frac{(y-y')^2}{2z^2}\right] d\alpha d\beta.$$  

The response function corresponds to a spherical wave in paraxial approximation. Similar to Huygen’s principle, where from each point in the object plane a spherical wave is emitted towards the image plane, here paraxial approximations of spherical waves are emitted.

To sum up, in position space paraxial beam propagation is given by:

$$u_f(x, y, z) = -\frac{ik}{2\pi} \exp[i(kz)] \int_{-\infty}^{\infty} u_0(x', y') \exp\left[i \frac{k}{2z} \left((x-x')^2 + (y-y')^2\right)\right] dx'dy'.$$

Of course, the two descriptions in position space and in the spatial Fourier domain are completely equivalent.
**The correspondence between real and frequency space**

![Diagram showing the correspondence between real and frequency space]

Relation between transfer and response function:

\[ h(x, y; z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} H(\alpha, \beta; z) \exp\left[i(\alpha x + \beta y)\right] d\alpha d\beta \]

Transfer functions for homogeneous space

\[ H(\alpha, \beta; z) = \exp\left[i\gamma(\alpha, \beta)z\right] = \exp\left[i\sqrt{k^2 - \alpha^2 - \beta^2} z\right] \text{ exact solution} \]

\[ H_0(\alpha, \beta; z) = \exp[i\alpha z] \exp\left[i\frac{\alpha^2 + \beta^2}{2k} z\right] \text{ Fresnel approximation} \]

with \( k = k(\omega) = \frac{\omega}{c}n(\omega) \)

**Remark on the validity of the scalar approximation**

\[ \mathbf{E}(\mathbf{r}, \omega) = \int \mathbf{E}(\alpha, \beta, \omega) e^{i(\alpha x + \beta y + \gamma z)} d\alpha d\beta \]

\[ \text{div} \mathbf{E}(\mathbf{r}, \omega) = 0 \rightarrow \alpha \hat{\mathbf{E}}_x + \beta \hat{\mathbf{E}}_y + \gamma \hat{\mathbf{E}}_z = 0 \]

A) One-dimensional beams

- Translational invariance in y-direction: \( \beta = 0 \)
- And linear polarization in y-direction: \( \hat{\mathbf{E}}_y \rightarrow U \)

→ Scalar approximation is exact since divergence condition is strictly fulfilled

B) Two-dimensional beams

- Finite beam which is localized in the x,y-plane: \( \alpha, \beta \neq 0 \)
- And linear polarization, w.l.o.g. in y-direction: \( \hat{\mathbf{E}}_x = 0, \hat{\mathbf{E}}_y \rightarrow U \)

→ Divergence condition: \( \beta \hat{\mathbf{E}}_y + \gamma \hat{\mathbf{E}}_z = 0 \)

\[ \hat{\mathbf{E}}_z(\alpha, \beta, \omega) = -\frac{\beta}{\gamma} \hat{\mathbf{E}}_y(\alpha, \beta, \omega) = -\frac{\beta}{\sqrt{k^2 - \alpha^2 - \beta^2}} \hat{\mathbf{E}}_y(\alpha, \beta, \omega) \approx 0 \]
In paraxial approximation \((\alpha^2 + \beta^2 \ll k^2)\) the scalar approximation is automatically justified.

### 2.7.3 The paraxial wave equation

In paraxial approximation the propagated spectrum is given by

\[
U_p(\alpha, \beta; z) = H_p(\alpha, \beta; z) U_0(\alpha, \beta)
\]

\[
= \exp(ikz) \exp\left(-\frac{i}{2k} \left(\frac{\alpha^2 + \beta^2}{z^2}\right)\right) U_0(\alpha, \beta)
\]

Let us introduce the slowly varying spectrum \(V(\alpha, \beta; z)\):

\[
\Rightarrow U_p(\alpha, \beta; z) = \exp(ikz)V(\alpha, \beta; z) \Rightarrow V(\alpha, \beta; z) = \exp\left(-\frac{i}{2k} \left(\frac{\alpha^2 + \beta^2}{z}\right)\right) V_0(\alpha, \beta).
\]

Differentiation of \(V\) with respect to \(z\) gives:

\[
\frac{i}{c} \frac{\partial}{\partial z} V(\alpha, \beta; z) = \frac{1}{2k} \left(\alpha^2 + \beta^2\right) V(\alpha, \beta; z)
\]

Fourier transformation back to position space leads to the so-called paraxial wave equation:

\[
\frac{i}{c} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} V(\alpha, \beta; z) \exp\left[\frac{i}{2} (\alpha x + \beta y)\right] \alpha \beta \, d\alpha d\beta
\]

\[
= \frac{1}{2k} \int_{-\infty}^{\infty} \left(\alpha^2 + \beta^2\right) V(\alpha, \beta; z) \exp\left[\frac{i}{2} (\alpha x + \beta y)\right] \alpha \beta \, d\alpha d\beta
\]

\[
\Rightarrow \frac{i}{c} \frac{\partial}{\partial z} v(x, y, z) = \frac{1}{2k} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \int_{-\infty}^{\infty} V(\alpha, \beta; z) \exp\left[\frac{i}{2} (\alpha x + \beta y)\right] \alpha \beta \, d\alpha d\beta
\]

\[
\Rightarrow \frac{i}{c} \frac{\partial}{\partial z} v(x, y, z) + \frac{1}{2k} \Delta v(x, y, z) = 0 \quad \text{paraxial wave equation}
\]

The slowly varying envelope \(v(x, y, z)\) (Fourier transform of the slowly varying spectrum) relates to the scalar field as \(u_p(x, y, z) = v(x, y, z) \exp(ikz)\).

### Extension of the wave equation to weakly inhomogeneous media

(slowly varying envelope approximation - SVEA)

There is an alternative, more general way to derive the paraxial wave equation, the so-called slowly varying envelope approximation. This approximation even allows us to treat inhomogeneous media. We will include inhomogeneous media in this derivation even though the current chapter of this lecture is devoted to inhomogeneous media.

We start from the scalar Helmholtz equation. However, we should mention that extrapolating the discussion on the scalar approximation above towards inhomogeneous media this is already an approximation assuming weak spatial fluctuations in \(\varepsilon(r, \omega)\).

\[
\Delta u(x, y, z) + k^2(\varepsilon(r, \omega)) u(x, y, z) = 0 \quad \text{with} \quad k^2(\varepsilon(r, \omega)) = \frac{\varepsilon^2(r, \omega)}{\varepsilon(r, \omega)}
\]

We use the ansatz \(u(x, y, z) = v(x, y, z) \exp(ikz)\) with \(\tilde{k} = \langle k \rangle\) being the average wavenumber. With the SVEA condition

\[
\left|\tilde{k} \right| \gg |\tilde{\varepsilon} v / \tilde{\varepsilon}z|
\]

we can simplify the scalar Helmholtz equation as follows:

\[
\frac{\partial^2}{\partial z^2} v(x, y, z) + 2i k_0 \frac{\partial}{\partial z} v(x, y, z) + \Delta^2 v(x, y, z) + \left[\tilde{k}^2(r, \omega) - k^2\right] v(x, y, z) = 0
\]

\[
\Rightarrow \frac{\partial^2}{\partial z^2} v(x, y, z) + \frac{1}{2k_0} \Delta^2 v(x, y, z) + \left[\frac{k^2(r, \omega) - \tilde{k}^2}{2k}\right] v(x, y, z) = 0
\]

This is the paraxial wave equation for inhomogeneous media (weak index contrast).

### 2.8 Propagation of Gaussian beams

The propagation of Gaussian beams is an important special case. First of all, the transversal fundamental mode of many lasers has Gaussian shape. Second, in paraxial approximation it is possible to compute the Gaussian beam evolution analytically.

![Fundamental Gaussian beam in focus](image)

The general form of a Gaussian beam is elliptic, with curved phase.

\[
u_0(x, y) = v_0(x, y) = A_0 \exp\left[-\frac{x^2}{w_0^2} - \frac{y^2}{w_0^2}\right] \exp\left[i \phi(x, y)\right]
\]

Here, we will restrict ourselves to rotational symmetry \(w_0^2 = w_0^2 = w_0^2\) and (initially) ‘flat’ phase \(\phi(x, y) = 0\), which corresponds to a beam in the focus. The Gaussian beam in the focal plane (flat phase) is characterized by
amplitude $A$ and width $w_0$: $u_0(x^2 + y^2) = A_0 \exp(-1) = A_0 / e$. In practice, the so-called 'full width at half maximum' (FWHM) is often used instead of $w_0$.

$$u_0(x^2 + y^2) = \exp\left(-\frac{w^2_{\rm FWHM}}{2w_0^2}\right) \approx \frac{1}{2}$$

$$\frac{w^2_{\rm FWHM}}{2w_0^2} = -\ln 2 \Rightarrow w^2_{\rm FWHM} = 2 \ln 2 w_0^2 \approx 1.386 w_0^2$$

2.8.1 Propagation in paraxial approximation

Let us now compute the propagation of a Gaussian beam starting from the focus in paraxial approximation:

1) Field at $z=0$:

$$u_0(x, y) = v_0(x, y) = A_0 \exp\left(-\frac{x^2 + y^2}{w_0^2}\right).$$

2) Angular spectrum at $z=0$:

$$U_0(\alpha, \beta) = V_0(\alpha, \beta) = \frac{1}{(2\pi)^2} A_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2 + y^2}{w_0^2}\right] \exp[-i(\alpha x + \beta y)] dx dy$$

$$= A_0 \frac{w_0^2}{4\pi} \exp\left(-\frac{\alpha^2 + \beta^2}{4/w_0^2}\right) = A_0 \frac{w_0^2}{4\pi} \exp\left(-\frac{\alpha^2 + \beta^2}{w_0^2}\right).$$

We see that the angular spectrum has a Gaussian profile as well and that the width in position space and Fourier space are linked by $w_x \times w_\beta = 2$

![Angular spectrum in the focal plane](image)

C) Check if paraxial approximation is fulfilled:

We can say that $U_0(\alpha, \beta) \approx 0$ for $(\alpha^2 + \beta^2) \geq 16 / w_0^2$, because $\exp(-4) \approx 0.02$.

For paraxial approximation we need $k^2 \gg (\alpha^2 + \beta^2)$

$$\Rightarrow k^2 \gg 16 / w_0^2$$

$$\Rightarrow w_0^2 \gg \frac{16}{(2\pi)^2 n} \approx \left(\frac{2\lambda}{\pi n}\right)^2 \approx \left(\frac{\lambda}{n}\right)^2,$$

$$\Rightarrow \text{paraxial approximation works for } w_0 \gg 10 \frac{\lambda}{n} = 10 \lambda_n$$

D) Propagation of the angular spectrum:

$$U(\alpha, \beta; z) = V(\alpha, \beta; z) \exp(\pm i k z)$$

$$V(\alpha, \beta; z) = U_0(\alpha, \beta) \exp\left[-\frac{1}{2k} (\alpha^2 + \beta^2 - 2k z)\right]$$

$$= A_0 \frac{w_0^2}{4\pi} \exp\left[-\frac{\alpha^2 + \beta^2}{4} \frac{1}{2k} \exp\left(-\frac{\alpha^2 + \beta^2}{2k z}\right)\right].$$

E) Fourier back-transformation to position space

$$v(x, y, z) = A_0 \frac{w_0^2}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x^2 + y^2}{w_0^2}\right) + i(\alpha x + \beta y)\right] d\alpha d\beta$$

$$= A_0 \frac{1}{1 + \frac{2k z}{kw_0^2}} \left[\frac{x^2 + y^2}{w_0^2 (1 + 2k z / kw_0^2)}\right].$$

With the Rayleigh length $z_0$ which determines the propagation of a Gaussian beam:

$$z_0 = \frac{kw_0^2}{2} = \frac{\pi}{\lambda_n} \approx \frac{1}{n}$$

Note that we use the slowly varying envelope $v$!

Conclusion:

- Gaussian beam keeps its shape, but amplitude, width, and phase change upon propagation
- Two important parameters: propagation length $z$ and Rayleigh length $z_0$

Some books use the “diffraction length” $L_n = 2z_0$, a measure for the “focus depth” of the Gaussian beam. E.g.: $w_0 \approx 10 \lambda_n \Rightarrow L_n \approx 600 \lambda_n$. 
From our computation above we know that the Gaussian beam evolves like:

$$v(x, y, z) = A(z) \frac{1}{1 + i z / z_0} \exp \left[ -\frac{x^2 + y^2}{w_0^2 \left( 1 + i z / z_0 \right)} \right]$$

For practical use, we can write this expression in terms of z-dependent amplitude, width, etc.:

$$v(x, y, z) = A_h \frac{1 - i z / z_0}{1 + (z / z_0)^2} \exp \left[ -\frac{x^2 + y^2}{w_0^2 \left( 1 + (z / z_0)^2 \right)} \right] \exp \left[ \frac{i}{2} \frac{k}{z} \frac{(x^2 + y^2)}{1 + (z / z_0)^2} \right] \exp [i \phi(z)].$$

Here we used that $w_0^2 = 2z_0 / k$. The $(x,y)$-independent phase $\phi(z)$ is given by $\tan \phi = -z / z_0$, the so-called Gouy phase shift.

In conclusion, we see that the propagation of a Gaussian beam is given by a z-dependent amplitude, width, phase curvature and phase shift:

$$v(x, y, z) = A(z) \exp \left[ -\frac{x^2 + y^2}{w^2(z)} \right] \exp \left[ \frac{i}{2} \frac{k}{R(z)} \left( x^2 + y^2 \right) \right] \exp [i \phi(z)].$$

Discussion

The amplitude is given as:

$$A(z) = A_h \frac{1}{1 + \frac{z}{z_0}} = A_h \frac{1}{\sqrt{1 + \left( \frac{z}{z_0} \right)^2}} \frac{1}{\sqrt{1 + \left( \frac{2z}{I_0} \right)^2}}.$$  

Hence, we get for the intensity profile $I \sim A^2$:

The on-axis intensity $(x=y=0)$ evolves like:

$$w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} = w_0 \sqrt{1 + \left( \frac{2z}{I_0} \right)^2}.$$
The beam radius $W(z)$ has its minimum value $W_0$ at the waist ($z = 0$), reaches $\sqrt{2}W_0$ at $z = \pm z_0$, and increases linearly with $z$ for large $z$.

The radius of the phase curvature is given by

$$R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] = z \left[ 1 + \left( \frac{L_0}{2z} \right)^2 \right]$$

The flat phase in the focus ($z=0$) corresponds to an infinite radius of curvature. The strongest curvature (minimum radius) appears at the Rayleigh distance from the focus. The $(x,y)$-independent Gouy phase is given by

$$\Phi(x,y,z) = \left( k \frac{x^2 + y^2}{2R(z)} + \varphi(z) \right) = \text{const.}$$

2.8.2 Propagation of Gauss beams with q-parameter formalism

In the previous chapter we gave the expressions for Gaussian beam propagation, i.e., we know how amplitude, width, and phase change with the propagation variable $z$. However, the complex beam parameter $q(z)$ allows an even simpler computation of the evolution of a Gaussian beam. In fact, if we take the inverse of the "q-parameter",

$$q(z) = z - \frac{1}{z_0}$$

we can observe that real and imaginary part are directly linked to radius of phase curvature and beam width:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{1}{\lambda_n w_0(z)} = \frac{1}{z - \frac{1}{z_0}} = \frac{z}{z^2 + z_0^2} + \frac{z_0}{z^2 + z_0^2} = \frac{1}{z(1 + \frac{z_0^2}{z^2})} + \frac{1}{z_0(1 + \frac{z_0^2}{z^2})}$$

Thus, the q-parameter describes beam propagation for all $z$!

Example: propagation in homogeneous space by $z = d$

A) initial conditions:

$$\frac{1}{q(0)} = \frac{1}{R(0)} + \frac{\lambda_n}{\pi w_0(0)}$$

B) propagation (by definition of q parameter) $q(d) = q(0) + d$

C) q-parameter at $z = d$ determines new width and radius of curvature

$$\frac{1}{q(d)} = \frac{1}{q(0) + d} \pm \frac{1}{R(d)} \frac{\lambda_n}{\pi w_0^2(d)}$$
2.8.3 Gaussian optics

We have seen in the previous chapter that the complex q-parameter formalism makes a simple description of beam propagation possible. The question is whether it is possible to treat optical elements (lens, mirror, etc.) as well.

**Aim:** for given \( R_0, w_0 \) (i.e. \( q_0 \)) \( \rightarrow R_n, w_n \) (i.e. \( q_n \)) after passing through \( n \) optical elements

We will evaluate the q-parameter at certain propagation distances, i.e., we will have values at discrete positions: \( q(z_i) \rightarrow \hat{q} \).

Surprising property: We can use ABCD-Matrices from ray optics! This is remarkable because here we are doing wave-optics (but with Gaussian beams).

How did it work in geometrical optics?

A) propagation through one optical element:

\[
\hat{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

B) propagation through multiple elements:

\[
\hat{M} = \hat{M}_n \hat{M}_{n-1} \cdots \hat{M}_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

C) matrix connects distances to the optical axis \( y \) and inclination angles \( \Theta \) before and after the element

\[
\begin{bmatrix} y_2 \\ \Theta_2 \end{bmatrix} = \hat{M} \begin{bmatrix} y_1 \\ \Theta_1 \end{bmatrix}.
\]

**Link to Gaussian beams**

Let us consider the distance to the intersection of the ray with the optical axis, as it was defined in chapter 1.6.1 on "The ray-transfer-matrix":

\[
z_1 = \frac{y_1}{\Theta_1} \Rightarrow z_2 = \frac{y_2}{\Theta_2} = \frac{A y_1 + B \Theta_1}{C y_1 + D \Theta_1} = \frac{A z_1 + B}{C z_1 + D}
\]

The distances \( z_1, z_2 \) are connected by matrix elements, but not by normal matrix vector multiplication.

It turns out that we can pass to Gaussian optics by replacing \( z \) by the complex beam parameter \( q \). The propagation of \( q \)-parameters through an optical element is given by:

\[
q_i = \frac{A q_0 + B}{C q_0 + D}
\]

\( \rightarrow \) propagation through \( N \) elements:

\[
q_n = \frac{A q_0 + B}{C q_0 + D}
\]

with the matrix \( \hat{M} = \hat{M}_n \hat{M}_{n-1} \cdots \hat{M}_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \).
This works for all ABCD matrices given in chapter 1.6 for ray optics!!!

Here: we will check two important examples:

i) propagation in free space by $z = d$ :
   
   $q(d) = q(0) + d$
   
   $M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
   
   $q_i = \frac{A q_0 + B}{C q_0 + D} = q_0 + d$

ii) thin lens with focal length $f$

   What does a thin lens do to a Gaussian beam $\exp \left( -\frac{(x^2 + y^2)}{w_0^2} \right)$ in paraxial approximation?
   
   - no change of the width
   
   - but change of phase curvature $R_i$:
     
     $\exp \left( \frac{1}{2} \frac{k}{R_i} \left( \frac{x^2 + y^2}{R_i} \right) \right)$

   How can we see that?

   Trick:

   We start from the focus which is produced by the lens with $z_0 = z_f = \frac{\pi w_0^2}{\lambda}$

   and $w_f$ is the focal width. The radius of curvature evolves as:

   $R(z) = z \left[ 1 + \left( \frac{z_f}{z} \right)^2 \right] \approx z$ for $z \gg z_f$

   We can invert the propagation from the focal position to the lens at the distance of the focal length $f$ and obtain $R_i = -f$

   $f < 0 \quad f > 0 \quad M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

2.8.4 Gaussian modes in a resonator

In this chapter we will use our knowledge about paraxial Gaussian beam propagation to derive stability conditions for resonators. An optical cavity or optical resonator is an arrangement of mirrors that forms a standing wave cavity resonator for light waves. Optical cavities are a major component of lasers, surrounding the gain medium and providing feedback of the laser light (see He-Ne laser experiment in Labworks).

2.8.4.1 Transversal fundamental modes (rotational symmetry)
The general idea to get a **stable** light configuration in a resonator is that mirrors and wave fronts (planes of constant phase) coincide. Then, radiation patterns are reproduced on every round-trip of the light through the resonator. Those patterns are the so-called **modes** of the resonator.

In paraxial approximation and Gaussian beams this condition is easily fulfilled: The radii of mirror and wave front have to be identical!

In this lecture we use the following conventions (different to Labworks script, see remark below!):

- \( z_{1,2} \) is the position of mirror '1','2'; \( z=0 \) is the position of the focus!
- \( d \) is the distance between the two mirrors \( \rightarrow z_2 - z_1 = d \)
- because \( R(z) = z + \frac{z_2^2}{z} \rightarrow \) radius of wave front <0 for \( z <0 \)
- from Chapter 1: beam hits concave mirror \( \rightarrow \) radius \( R(i=1,2) <0 \).
- beam hits convex mirror \( \rightarrow \) radius \( R(i=1,2) >0 \).

Examples:

**A)\)** \( R(z_1), R(z_2) >0; R_1 >0, R_2 <0; z_1 >0, z_2 >0 \)

\[ R_1 = R(z_1), \quad R_2 = -R(z_2) \]

\[ \wedge \quad R_1 = z_1 + \frac{z_2^2}{z_1}, \quad -R_2 = z_2 + \frac{z_2^2}{z_2} \]

In both expressions we find the Rayleigh length \( z_0 \), which we eliminate: \( z_1(R_1 - z_1) = -z_2(R_2 + z_2) \)

with \( z_2 = z_1 + d \) we find \( z_1 = \frac{-d(R_1 + d)}{R_1 + R_2 + 2d} \).

Now we can choose \( R_1, R_2, d \) and compute modes in the resonator. However, we have to make sure that those modes exist. In our calculations above we have eliminated the Rayleigh length \( z_0 \), a real and positive quantity. Hence, we have to check that the so-called stability condition \( z_0^2 >0 \) is fulfilled!

\[ z_1^2 = R_1 z_1 - z_1^2 = -d(R_1 + d)(R_1 + d)(R_1 + R_2 + d) > 0 \]

The denominator \( (R_1 + R_2 + 2d)^2 \) is always positive we need to fulfill

\[ -d(R_1 + d)(R_1 + d)(R_1 + R_2 + d) > 0 \]

If we introduce the so-called resonator parameters

\[ g_1 = 1 + \frac{d}{R_1}, \quad g_2 = 1 + \frac{d}{R_2} \]

We can re-express the stability condition as

\[ -d(R_1 + d)(R_1 + d)(R_1 + R_2 + d) = d g_1 g_2 R_1 R_2 \frac{(1 - g_1 g_2) R_1 R_2}{d} = 0 \]

This inequality is fulfilled for

\[ 0 < g_1 g_2 < 1 \quad \text{or} \quad 0 < \left(1 + \frac{d}{R_1}\right) \left(1 + \frac{d}{R_2}\right) < 1 \]

This final form of the stability condition can be visualized: The range of stability of a resonator lies between coordinate axes and hyperbolas:
Resonator stability diagram. A spherical-mirror resonator is stable if the parameters $g_1 = 1 + d/R_1$ and $g_2 = 1 + d/R_2$ lie in the unshaded regions bounded by the lines $g_1 = 0$ and $g_2 = 0$, and the hyperbola $g_2 = 1/g_1$. $R$ is negative for a concave mirror and positive for a convex mirror. Various special configurations are indicated by letters. All symmetrical resonators lie along the line $g_1 = g_2$. 

Examples for a stable and an unstable resonator:

A) $R_1, R_2 < 0; |R_i| > d, |R_i| > d; \cap 0 \leq g_1 \leq 1, 0 \leq g_2 \leq 1; \cap 0 \leq g_1 g_2 \leq 1 \cap \text{stable}$

B) $R_1, R_2 < 0; |R_i| < d, |R_i| > d; \cap g_1 \leq 0, 0 \leq g_2 \leq 1; \cap g_1 g_2 \leq 0 \cap \text{unstable}$

Remark: connection to He-Ne-Labwork script (and Wikipedia):

In Labworks (he_ne_laser.pdf) a slightly different convention is used:
- Direction of z-axis reversed for the two mirrors
- beam hits concave mirror $\rightarrow$ radius $R_i = 1/R > d$
- beam hits convex mirror $\rightarrow$ radius $R_i = 1/R < d$
- $z_{i,2}$ is the distance of mirror ‘1’, ‘2’ to the focus!
- $d$ is the distance between the two mirrors $\rightarrow z_2 + z_1 = d$

Examples:

A) $R(z_1) < 0, R(z_2) > 0; R_1 < 0, R_2 > 0; z_1 < 0, z_2 > 0$

B) $R(z_1) > 0, R(z_2) > 0; R_1, R_2 > 0; z_1 > 0, z_2 > 0$

Then the conditions for stability are:
With analog calculation as above we find with for the resonator parameters

\[ g_1 = \left(1 - \frac{d}{R_1}\right), \quad g_2 = \left(1 - \frac{d}{R_2}\right) \]

the same stability condition

\[ g_1 g_2 \left(1 - g_1 g_2\right) \left(R_1 R_2\right)^2 > 0, \quad 0 < g_1 g_2 < 1. \]

### 2.8.4.2 Higher order resonator modes

For the derivation of the above stability condition we needed the wave fronts only. Hence, there may exist other modes with same wave fronts but different intensity distribution. For the fundamental mode we have:

\[ v_q(x, y, z) = A \frac{w_0}{w(z)} \exp \left[-x^2 + y^2 \frac{w(z)}{w^2}\right] \exp \left[\frac{1}{2} \left(\frac{k x^2 + y^2}{R(z)}\right)^2 \exp[1] \varphi(z)\right]. \]

Higher order modes: \((x, y)\)-dependence of phase is the same

\[ \begin{align*}
    u_{lm}(x, y, z) &= A_{lm} \frac{w_0}{w(z)} G_l \left[\sqrt{\frac{2}{w(z)}} \frac{x}{w(z)}\right] G_m \left[\sqrt{\frac{2}{w(z)}} \frac{y}{w(z)}\right] \times \\
    & \quad \exp \left[\frac{1}{2} \left(\frac{k x^2 + y^2}{R(z)}\right)^2 \exp[1] \varphi(x) \exp[i(l + m + 1)\varphi(z)]\right] \\
    G_l(\xi) &= H_l(\xi) \exp \left(-\frac{\xi^2}{2}\right) 
\end{align*} \]

The functions \(G_l\) are given by the so-called Hermite polynomials:

\[ H_l(\xi) \quad (H_0 = 1, H_1 = 2\xi \quad \text{and} \quad H_{j+1} = 2\xi H_j - 2H_{j-1}). \]

### 2.9 Dispersion of pulses in homogeneous isotropic media

#### 2.9.1 Pulses with finite transverse width (pulsed beams)

In the previous chapters we have treated propagation of monochromatic beams, where the frequency \(\omega\) was fix and therefore the wavenumber \(k(\omega)\) was constant as well. This is the typical situation when we deal with continuous-wave (cw) lasers.

However, for many applications (spectroscopy, nonlinear optics, telecommunication, material processing) we need to consider the propagation of pulses. In this situation, we have typical envelope length \(T_0\) of

\[ 10^{-13}\text{s}(100\text{fs}) \leq T_0 \leq 10^{-10}\text{s}(100\text{ps}). \]

Let us compute the spectrum of the (Gaussian) pulse:

\[ f(t) = \exp(-i\omega_0 t) \exp\left(-\frac{t^2}{T_0^2}\right) \]

\[ F(\omega) = \exp\left[-\frac{(\omega - \omega_0)^2}{4/T_0^2}\right] \rightarrow \omega_s^2 = \frac{4}{T_0^2} \rightarrow \omega_s T_0 = 2 \]

\[ \Rightarrow \text{spectrum width:} \quad 4 \cdot 10^{10}\text{s}^{-1} \leq \omega_s \leq 4 \cdot 10^{13}\text{s}^{-1} \]

\[ \Rightarrow \text{center frequency of visible light:} \quad \omega_0 = 2\pi v = 4 \cdot 10^{15}\text{s}^{-1} \]

\[ \Rightarrow \text{optical cycle:} \quad T_0 = 2\pi/\omega_0 \approx 1.6\ \text{fs} \]

Hence, we have the following order of magnitudes:

\[ \omega_s \ll \omega_0 \rightarrow \omega - \omega_0 = \omega_0 \ll \omega_s \]

In this situation it can be helpful to replace the complicated frequency dependence (dispersion relation) of the wave vector \(k(\omega)\) or the wave number \(k(\omega)\) by a Taylor expansion at the central frequency \(\omega = \omega_0\).
In most cases, a parabolic (or cubic) approximation of the frequency dependence in the dispersion relation will be sufficient:

\[ k(\omega) \approx k(\omega_0) + \frac{\partial k}{\partial \omega} (\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 k}{\partial \omega^2} (\omega - \omega_0)^2 + \ldots \]

The following terminology for the individual expansion coefficients is commonly used in the literature. It associates the physics, which is inherited in the dispersion relation, with the three parameters of the Taylor expansion.

**The three expansion coefficients and their physical significance**

**A) Phase velocity** \(v_p\)

\[ k(\omega_0) = k_0, \quad v_p = \frac{1}{c} \frac{k_0}{\omega_0} = \frac{n(\omega_0)}{c} \]

\(\rightarrow\) velocity of the phase front for the light at the central frequency \(\omega = \omega_0\)

**B) Group velocity or group index** \(v_g\)

\[ \frac{\partial k}{\partial \omega} \bigg|_{\omega_0} = \frac{1}{v_g} \]

\(\rightarrow\) group velocity is the velocity of the center of the pulse (see below)

\[ k(\omega) = \frac{\omega}{c} n(\omega) \rightarrow \frac{1}{v_g} = \frac{n(\omega_0)}{c} + \omega_0 \frac{\partial n}{\partial \omega} \bigg|_{\omega_0} \]

**C) Group velocity dispersion (GVD) or simply dispersion** \(D_\omega\)

\[ \frac{\partial^2 k}{\partial \omega^2} \bigg|_{\omega_0} = D_\omega \]

\(\rightarrow\) GVD changes pulse shape upon propagation (see below)

\[ D = D_\omega = \frac{\partial^2 k}{\partial \omega^2} \bigg|_{\omega_0} = \frac{\partial}{\partial \omega} \left( \frac{1}{v_g} \right) \]

\[ D = \frac{\partial}{\partial \omega} \left( \frac{1}{v_g} \right) = - \frac{1}{v_g^2} \frac{\partial v_g}{\partial \omega} \]

\(\rightarrow D > 0 \Leftrightarrow \frac{\partial v_g}{\partial \omega} < 0 \)

\(\rightarrow D < 0 \Leftrightarrow \frac{\partial v_g}{\partial \omega} > 0 \)

Alternatively in telecommunication one often uses

\[ D_\lambda = \frac{\partial}{\partial \lambda} \left( \frac{1}{v_g} \right) = - \frac{2\pi}{\lambda^2} c D_\omega \]

Let us now discuss the propagation of pulsed beams. We start with the scalar Helmholtz equation, with the full dispersion (no Taylor expansion yet):

\[ \nabla^2 \Psi(r, \omega) + \frac{\omega^2}{c^2} \varepsilon(\omega) \Psi(r, \omega) = 0 \]

In contrast to monochromatic beam propagation, we now have for each frequency \(\omega\) one Fourier component of the optical field:

\[ \text{dispersion relation: } k^2(\omega) = \frac{\omega^2}{c^2} \varepsilon(\omega) \]

Hence, we need to consider the propagation of the Fourier spectrum (Fourier transform in space and time):
\[
U(\alpha, \beta, \omega; z) = U_0(\alpha, \beta, \omega) \exp\left[ \frac{\gamma}{\omega} (\alpha, \beta, \omega) \right] z
\]

with \( \gamma(\alpha, \beta, \omega) = \sqrt{k^2(\omega) - \alpha^2 - \beta^2} \)

The initial spectrum at \( z = 0 \) is \( U_0(\alpha, \beta, \omega) \)

\[
U_0(\alpha, \beta, \omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0(x, y, t) \exp\left[ -i(\alpha x + \beta y - \omega t) \right] dx dy dt
\]

Let us further assume that the Fresnel (paraxial) approximation is justified \((k(\omega) >> \alpha^2 + \beta^2)\)

\[
U(\alpha, \beta, \omega; z) \approx U_0(\alpha, \beta, \omega) \exp\left[ i k(\omega) z \right] \exp\left[ -i \frac{\alpha^2 + \beta^2}{2k(\omega)} z \right]
\]

We see that propagation of pulsed beams in Fresnel approximation in Fourier space is described by the following propagation function (transfer function):

\[
H_p(\alpha, \beta, \omega; z) = \exp\left[ i k(\omega) z \right] \exp\left[ -i \frac{\alpha^2 + \beta^2}{2k(\omega)} z \right]
\]

Now let us consider the Taylor expansion of \( k(\omega) \) from above. If the pulse is not too short, we can replace the wave number \( k(\omega) \) by

\[
k(\omega) \approx k(\omega_0) + \frac{\partial k}{\partial \omega} (\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 k}{\partial \omega^2} (\omega - \omega_0)^2 + ...
\]

Moreover, in the second term \( \exp[-i(\alpha^2 + \beta^2)/2k(\omega)] \) of the transfer function (which is already small due to paraxiality) we can approximate the frequency dependence of the wave number by \( k(\omega) \approx k(\omega_0) = k_0 \). This approximation is sufficiently accurate to describe the diffraction of pulsed beams which are not too short. But this approximation would break down for \( T < 15 \) fs since for such short pulses the frequency spectrum would become very wide. By introducing this approximation, we obtain the so-called parabolic approximation:

\[
H_{p,F}(\alpha, \beta, \omega; z) \approx \exp\left[ i k_0 z \right] \exp\left[ i \frac{1}{v_g} (\omega - \omega_0) z \right] \times \exp\left[ -i \frac{\alpha^2 + \beta^2}{2k_0} z \right]
\]

\[
= \exp\left[ i k_0 z \right] \exp\left[ i z \left( \frac{\omega - \omega_0}{v_g} + \frac{1}{2} \frac{k_0}{v_g} (\alpha^2 + \beta^2) \right) \right]
\]

with \( \varnothing = \omega - \omega_0 \)

Based on the last line of the above equation we can introduced a new variant of the propagation function, where the frequency argument is replaced by the frequency difference \( \varnothing \) from the center frequency \( \omega_0 \)

\[
H_{p,F}(\alpha, \beta, \varnothing; z) \approx \exp\left[ i k_0 z \right] H_{p,F}(\alpha, \beta, \omega_0; z)
\]

The new transfer function \( H_{p,F}(\alpha, \beta, \omega_0; z) \) is the propagation function for the slowly varying envelope \( v(x, y, t) \):

\[
u(x, y, t) = \exp\left[ i k_0 z \right] \int_{-\infty}^{\infty} U_0(\alpha, \beta, \omega) H_{p,F}(\alpha, \beta, \omega_0, z) \times \exp\left[ i (\alpha x + \beta y - \omega_0 t) \right] d\omega d\beta d\omega
\]

Illustration of the slowly varying envelope in the spectral domain

\[
u(x, y, t) = \nu(x, y, t) \exp\left[ i (k_0 z - \omega_0 t) \right]
\]

In order to complete the formalism, we also need to define the initial spectrum of the slowly varying envelope
\( u_0(x, y, t) = v_0(x, y, t) \exp(-i\omega_0t) \)

\[ \rightarrow V_0(\alpha, \beta, \omega) = \frac{1}{(2\pi)} \int\int v_0(x, y, t) \exp[-i(\alpha x + \beta y - \omega_0t)] \, dx dy dt \]

Thus, the propagation of the slowly varying envelope is given by:

\[ v(x, y, z, t) = \int\int\int V_0(\alpha, \beta, \omega) \tilde{H}_\tau(\alpha, \beta, \omega; z) \exp[i(\alpha x + \beta y - \omega_0(t - z/v))] \, d\alpha d\beta d\omega \]

**Co-moving reference frame**

The next step is to introduce a co-moving reference frame with

\[ \tilde{H}(\alpha, \beta, \omega; z) = \exp\left[i\frac{\omega}{v_0}z\right] \tilde{H}_\tau(\alpha, \beta, \omega; z) \]

\[ v(x, y, z, t) = \int\int\int V_0(\alpha, \beta, \omega) \tilde{H}_\tau(\alpha, \beta, \omega; z) \exp\left[i(\alpha x + \beta y - \omega_0(t - z/v_0))\right] \, d\alpha d\beta d\omega \]

The last line above involves the propagation function \( \tilde{H}_\tau(\alpha, \beta, \omega; z) \), the propagation function for the slowly varying envelope in the co-moving frame of the pulse:

\[ \tau = t - \frac{z}{v_0} \]

This frame is called co-moving because \( \tilde{H}_\tau(\alpha, \beta, \omega; z) \) is now purely quadratic in \( \tilde{\omega} \), i.e., the pulse does not "move" anymore. In contrast, the linear \( \omega \)-dependence in Fourier space had given a shift in the time domain. Thus, the slowly varying envelope in the co-moving frame evolves as:

\[ \tilde{v}(x, y, z, \tau) = \int\int\int V_0(\alpha, \beta, \omega) \exp\left[i\frac{\omega}{v_0}z\right] \exp\left[i\left(\alpha x + \beta y - \omega_0\tau\right)\right] \, d\alpha d\beta d\omega \]

The optical field \( u \) reads in the co-moving frame as:

\[ \tilde{u}(x, y, z, \tau) = \tilde{v}(x, y, z, \tau) \exp\left[i\left(k_0z - \omega_0\tau\right)\right] = \tilde{v}(x, y, z, \tau) \exp\left[i\left(k_0z - \omega_0\frac{z}{v_0} - \omega_0\tau\right)\right] \]

**Propagation equation in real space**

Finally, let us derive the propagation equation for the slowly varying envelope in the co-moving frame. We start from the transfer function

\[ \tilde{V}(\alpha, \beta, \omega, z) = V_0(\alpha, \beta, \omega) \exp\left[i\frac{z}{2\omega_0}\left(D\omega^2 - \frac{\alpha^2 + \beta^2}{k_0}\right)\right] \]

Then we take the spatial derivative of the transfer function long the propagation direction \( z \)

\[ \frac{D\tilde{v}(x, y, z, \tau)}{Dz} = -\frac{1}{2} \left( D\omega^2 - \frac{\alpha^2 + \beta^2}{k_0} \right) \tilde{v}(x, y, z, \tau) \]

As before in the case of monochromatic beams, we use Fourier back-transformation to get the differential equation in the time-position domain

\[ \frac{D\tilde{v}(x, y, z, \tau)}{Dz} - D\frac{D^2}{D\tau^2} \tilde{v}(x, y, z, \tau) + \frac{1}{2k_0} \Delta^{(2)} \tilde{v}(x, y, z, \tau) = 0 \]

This is the scalar paraxial equation for propagation of so-called pulsed beams.

**Comment: Extension to inhomogeneous media**

By using the slowly varying envelope approximation, it is possible to generalize the scalar paraxial equation also for inhomogeneous media, when a weak index contrast is assumed.

\[ \frac{D\tilde{v}(x, y, z, \tau)}{Dz} - D\frac{D^2}{D\tau^2} \tilde{v}(x, y, z, \tau) + \frac{1}{2k_0} \Delta^{(2)} \tilde{v}(x, y, z, \tau) + \frac{k_0^2(r) - k_0^2}{2k_0} \tilde{v}(x, y, z, \tau) = 0 \]

with \( k_0 \approx \langle k_0(r) \rangle \)

For \( D = 0 \) the equation would be reduced to simple diffraction, as in the beam propagation scheme which was derived earlier.