2015-04-23

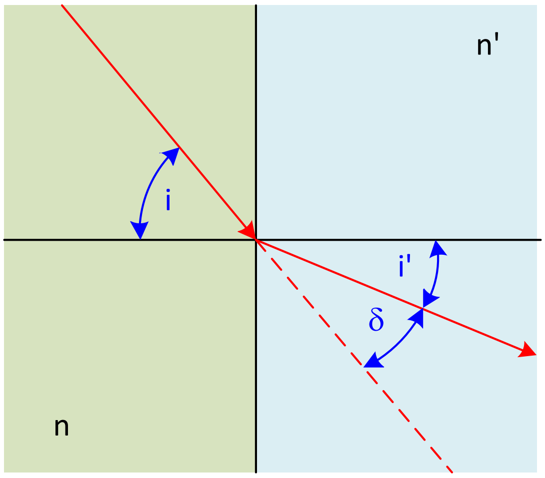
Prof. Herbert Gross
Yi Zhong, Chang Liu
Friedrich Schiller University Jena
Institute of Applied Physics
Albert-Einstein-Str 15
07745 Jena

Exercise Solutions:
Design and Correction of Optical Systems – Part 1

Exercise 1-1: Refraction angle deviation

The angle deviation δ at a plane surface between two media with refractive indices n and n' can be written as

$$\sin\frac{\delta}{2} = \frac{n'-n}{2n}\cdot\frac{\sin i'}{\cos\frac{i+i'}{2}}$$



1a) Derive this formula.
1b) Derive a formula for δ as a function of n, n' and I alone
1c) Evaluate the paraxial small angle limit of this formula

Solution:
1a)
Re-arrangement of the formula:

$$\sin\frac{\delta}{2} = \frac{n'-n}{2n}\cdot\frac{\sin i'}{\cos\frac{i+i'}{2}}$$

$$\sin\frac{\delta}{2}\cdot\cos\frac{i+i'}{2} = \frac{n'-n}{2n}\cdot\sin i'$$

Left side of this equation with $\delta = i - i'$

$$\sin\frac{\delta}{2}\cdot\cos\frac{i+i'}{2} = \sin\frac{i-i'}{2}\cdot\cos\frac{i+i'}{2}$$

$$= \left(\sin\frac{i}{2}\cos\frac{i'}{2} - \sin\frac{i'}{2}\cos\frac{i}{2}\right)\cdot\left(\cos\frac{i}{2}\cos\frac{i'}{2} - \sin\frac{i'}{2}\sin\frac{i}{2}\right)$$

$$= \sin\frac{i}{2}\cos\frac{i}{2}\cos^2\frac{i'}{2} + \sin\frac{i}{2}\cos\frac{i}{2}\sin^2\frac{i'}{2} - \sin\frac{i'}{2}\cos\frac{i'}{2}\cos^2\frac{i}{2} + \sin\frac{i'}{2}\cos\frac{i'}{2}\sin^2\frac{i}{2}$$

$$= \sin\frac{i}{2}\cos\frac{i}{2} - \sin\frac{i'}{2}\cos\frac{i'}{2} = \frac{1}{2}\sin i - \frac{1}{2}\sin i' = \frac{1}{2}\sin i - \frac{n}{2n'}\sin i = \frac{n'-n}{2n}\sin i$$

1b) From the drawing we see:

$$i - \delta = i'$$

$$\sin(i-\delta) = \sin i' = \frac{n}{n'}\sin i$$

$$\sin i\cos\delta - \sin\delta\cos i = \frac{n}{n'}\sin i$$

$$\sin i\cdot\left(\frac{n}{n'} - \cos\delta\right) = -\cos i\sin\delta$$

$$\sin^2 i\cdot\left(\frac{n^2}{n'^2} - \frac{2n}{n'}\cos\delta + \cos^2\delta\right) = \cos^2 i\cdot\left(1-\cos^2\delta\right)$$

$$\cos^2\delta - \frac{2n}{n'}\sin^2 i\cos\delta + \sin^2 i\cdot\left(1+\frac{n^2}{n'^2}\right) - 1 = 0$$

Quadratic equation for cosδ has the solution

$$\cos\delta = \frac{n}{n'}\sin^2 i + \sqrt{1+\frac{n^2}{n'^2}\sin^4 i - \left(1+\frac{n^2}{n'^2}\right)\sin^2 i}$$

1c)

For small angles, the sin(i) is a small quantity and is approximated by its argument. The root is expanded as a Taylor expansion in the form

$$\cos\delta = \frac{n}{n'}\sin^2 i + \sqrt{1+\frac{n^2}{n'^2}\sin^4 i - \left(1+\frac{n^2}{n'^2}\right)\sin^2 i} \approx \frac{n}{n'}i^2 + \sqrt{1-\left(1+\frac{n^2}{n'^2}\right)i^2}$$

$$\approx \frac{n}{n'}i^2 + \left[1-\frac{1}{2}\left(1+\frac{n^2}{n'^2}\right)i^2\right] = 1 + i^2\cdot\left(\frac{n}{n'} - \frac{1}{2} - \frac{n^2}{2n'^2}\right) = 1 + i^2\cdot\frac{2nn'-n'^2-n^2}{2n'^2} = 1 - i^2\cdot\frac{(n'-n)^2}{2n'^2}$$

With the expansion of the cos-function $\cos\delta = 1 - \frac{1}{2}\delta^2$

We get by comparing the quadratic terms

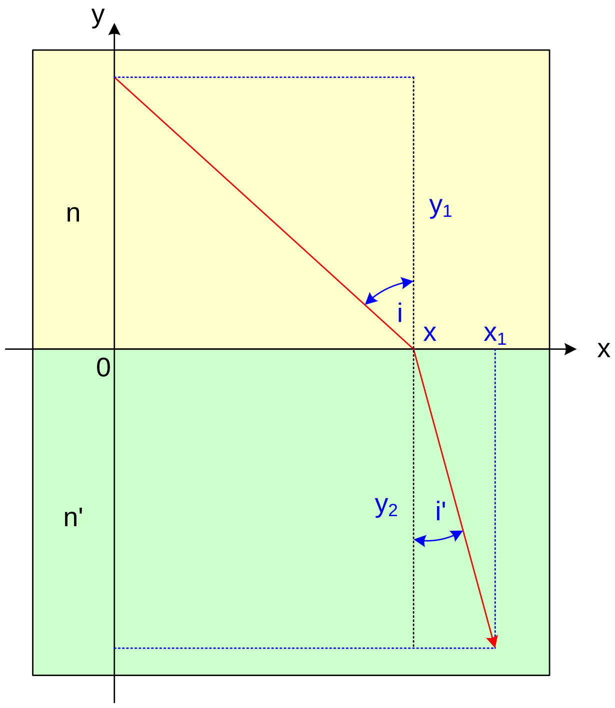
$$1 - \frac{1}{2}\delta^2 = 1 - i^2\cdot\frac{(n'-n)^2}{2n'^2}$$

$$\delta = i\cdot\frac{n'-n}{n'} = i\cdot\left(1-\frac{n}{n'}\right)$$

$$\delta = i - i'$$

Exercise 1-2: Law of Refraction by Fermat Principle

Derive the law of refraction by the Fermat principle of the smallest optical path length between two points above and below a plane interface. It is sufficient to consider only two dimensions.



Solution:

The optical path length as a measure of the action is given by

$$s_{OPL} = n \cdot \sqrt{y_1^2 + x^2} + n' \cdot \sqrt{y_2^2 + (x_1 - x)^2}$$

The path length should have a minimal value, if the correct refraction location x of the ray is realized. This corresponds to

$$\frac{\partial s_{OPL}}{\partial x} = n \cdot \frac{2x}{2\sqrt{y_1^2 + x^2}} + n' \cdot \frac{-(x_1 - x)}{\sqrt{y_2^2 + (x_1 - x)^2}} = 0$$

with the incidence angles

$$\sin i = \frac{x}{\sqrt{y_1^2 + x^2}} \quad , \quad \sin i' = \frac{x_1 - x}{\sqrt{y_2^2 + (x_1 - x)^2}}$$

we get the law of refraction

$$n \cdot \sin i = n' \cdot \sin i'$$

Exercise 1-3: Sellmeier Dispersion Formula

The measured refractive indices of the glass BK7 are given by the following data:

```
0.3650100        1.5362680
0.4046600        1.5302390
0.4358300        1.5266850
0.4800000        1.5228290
0.4861300        1.5223760
0.5460700        1.5187220
0.5875600        1.5168000
0.5892900        1.5167280
0.6328000        1.5150890
0.6438500        1.5147190
0.6562700        1.5143220
0.7065200        1.5128920
0.8521100        1.5098030
1.0139800        1.5073080
1.0600000        1.5066880
1.5296000        1.5009070
1.9701000        1.4949480
2.3254000        1.4892120
```

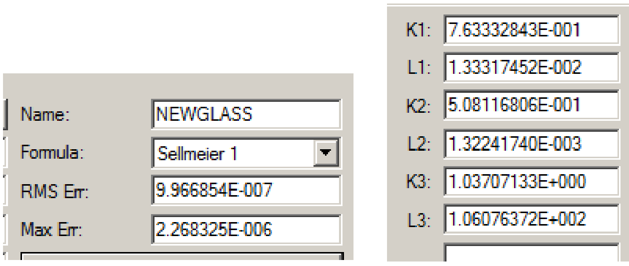
where the first column represents the wavelength in µm, the second column gives the corresponding indices.
Compute a numerical fit of these data by a 3-term Sellmeier dispersion formula. Check the model representation for the given wavelengths 0.40466, 0.54607, 0.6328, 0.70652, 1.01398 µm and the intermediate values λ=0.4, 0.5, 0.6, 0.7, 1.0 µm.
Discuss your results. Estimate the overall accuracy. Do you think, three Sellmeier terms fit the data well? What can be done to improve the results?

Solution 1 with Zemax:

Formula:

$$n^2 - 1 = \frac{K_1 \lambda^2}{\lambda^2 - L_1} + \frac{K_2 \lambda^2}{\lambda^2 - L_2} + \frac{K_3 \lambda^2}{\lambda^2 - L_3} .$$

Results:

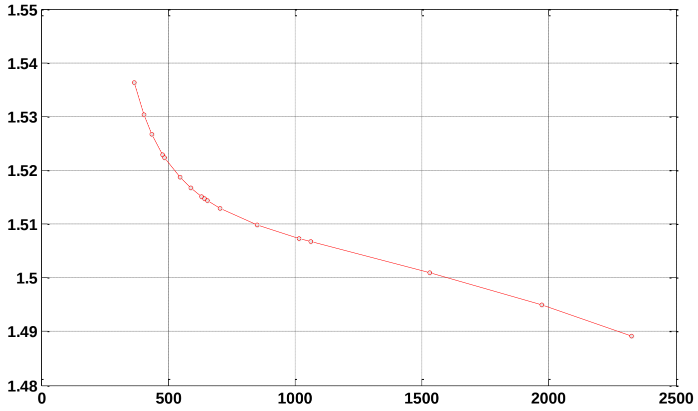


Interpolation:

```
    Glass     Temp     Pres     0.400000   0.500000   0.600000   0.700000   1.000000
              20.00    1.00   1.00000000 1.00000000 1.00000000 1.00000000 1.00000000
  BK7-FIT     20.00    1.00   1.53085027 1.52141416 1.51629377 1.51306353 1.50750352
              20.00    1.00   1.00000000 1.00000000 1.00000000 1.00000000 1.00000000
```

Glass	Temp	Pres	0.404660	0.546070	0.632800	0.706520	1.013980
	20.00	1.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
BK7-FIT	20.00	1.00	1.53024127	1.51872124	1.51508826	1.51289207	1.50730941
	20.00	1.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

Solution 2 with Matlab

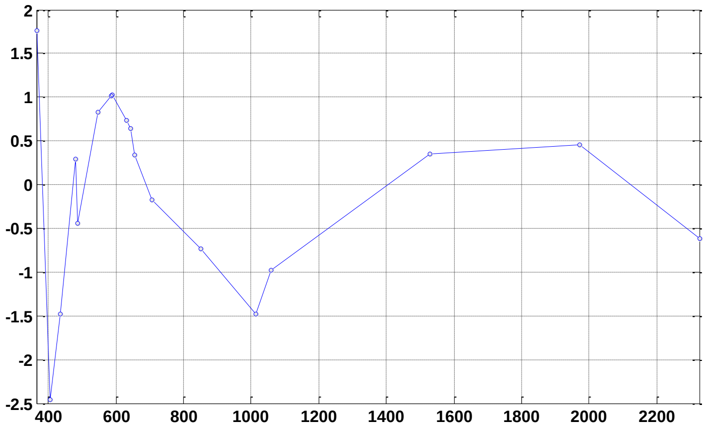


Result (the squared wavelength corresponds to the Zemax parameters L):

```
Max Res  : Rms  = 1.0523e-006
Param    : no   = 1.1911

Param    : K1    = 0.85283  wl1   = 0.11275  wlq   = 0.012712
Param    : K2    = 0.44569  wl2   = 8.5179   wlq   = 72.5553
Param    : K3    = 3.584    wl3   = 31.3498  wlq   = 982.8113
```

Residual values, scaled in 10^{-6}:



Interpolation results:

```
wl  = 0.4000000  n  = 1.5308505
wl  = 0.5000000  n  = 1.5214141
wl  = 0.6000000  n  = 1.5162937
wl  = 0.7000000  n  = 1.5130636
wl  = 1.0000000  n  = 1.5075036
wl  = 0.4046600  n  = 1.5302415
wl  = 0.5460700  n  = 1.5187212
```

```
wl  = 0.6328000  n  = 1.5150883
wl  = 0.7065200  n  = 1.5128922
wl  = 1.0139800  n  = 1.5073095
```

As it is seen, the deviations exhibits a systematic error. Therefore the description don't fit the behavior well. More terms in the Sellmeier equation could help. Alternatively, a truncatioin of the extrem wavelength delivers a better description in the inner region.
It is seen, that depending on the starting values, Zemax used two resonance point on the UV side while in Matlab two points on the IR side are found to be better. In the following table the single results are shown for comparison. The Matlab-result is a little bit better than Zemax. In the region from approximately 0.45 µm to 0,8 µm the interpolation is better than 1.e-6, which is sufficient for the most applications.

Comparison:

wavelength	original	fit Zemax	fit matlab	err zemax 10-6	err matlab 10-6	diff Zemax-Matlab 10-6
0,40466	1,53023900	1,53024127	1,53024150	2,500	2,270	-0,230
0,54607	1,51872200	1,51872124	1,51872120	-0,800	-0,760	0,040
0,63280	1,51508900	1,51508826	1,51508830	-0,700	-0,740	-0,040
0,70652	1,51289200	1,51289207	1,51289220	0,200	0,070	-0,130
1,01398	1,50730800	1,50730941	1,50730950	1,500	1,410	-0,090
0,40000		1,53085027	1,53085050			-0,230
0,50000		1,52141416	1,52141410			0,060
0,60000		1,51629377	1,51629370			0,070
0,70000		1,51306353	1,51306360			-0,070
1,00000		1,50750352	1,50750360			-0,080

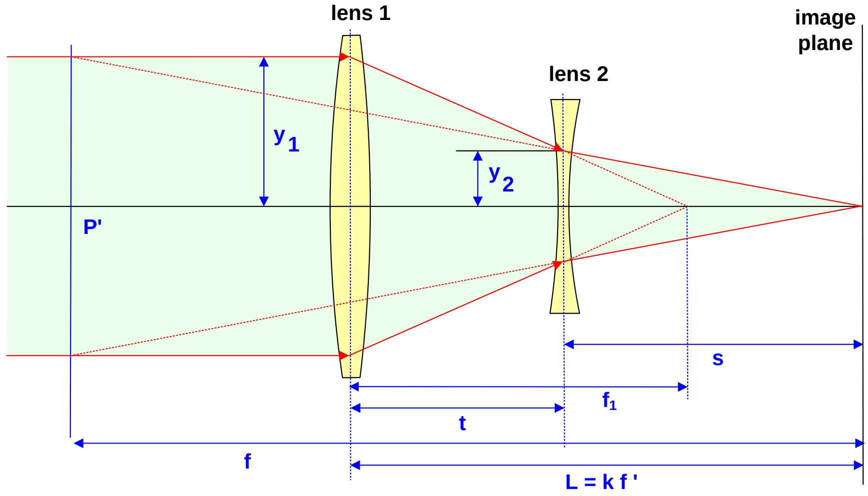
Exercise 1-4: Design of a Telephoto System

If a system combines a positive and a negative lens in a distance t, it is possible to move the principal plane in front of the system. The focal length f of the system is then larger than the overall size L. The so called telephoto factor is the ratio between overall length of the system (first lens until image plane) to the focal length $k = \frac{L}{f}$.

Calculate the individual focal lengths of both lenses as a function of given f, k and distance t between the two lenses.

Discuss the theoretical and a practical ranges of validity of these equations. What is a useful range for k in practice?

(Hint: $tan(u) \approx u$)



Solution:

The angle of an incoming axis-parallel ray with height y_1 behind the first lens is given by

$$u'_1 = -\frac{y_1}{f_1} \tag{1}$$

The ray height at the second lens is then

$$y_2 = y_1 - t \cdot u'_1 = y_1 \cdot \left(1 - \frac{t}{f_1}\right) \tag{2}$$

The ray angle behind the system is u'_2. It further defines the focal length by

$$u'_2 = -\frac{y_2}{s} = -\frac{y_1}{f} \tag{3}$$

If the ratio of the two ray heights is extracted from the equations (1) and (2) and is equated, we get

$$\frac{y_2}{y_1} = \frac{s}{f} = 1 - \frac{t}{f_1} \tag{4}$$

From this equation we get the first focal length

$$f_1 = \frac{t \cdot f}{t + f \cdot (1-k)} \tag{5}$$

The lens equation of the second lens can be written as

$$
\begin{aligned}
-\frac{1}{s_2}+\frac{1}{s'_2}&=\frac{1}{f_2}\\
-\frac{1}{f_1-t}+\frac{1}{s}&=\frac{1}{f_2}
\end{aligned}
\tag{6}
$$

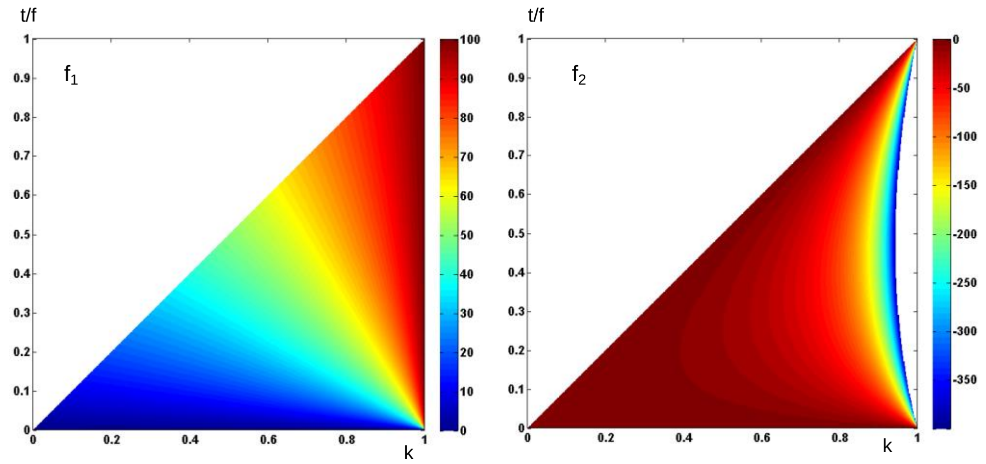
from this relation we get the second focal length as

$$
f_2=\frac{(f_1-t)\cdot(k\cdot f)-t}{f_1-k\cdot f}=-\frac{t}{f}\cdot\frac{k\cdot f-t}{1-k}
\tag{7}
$$

The telephoto factor is always in the range k = 0...1. two theoretical limits are the conditions:

1. s > 0 to get a real image
2. t<f to really get a telephoto effect

If for example f=100 mm is assumed we get the following figures for the values of f_1 and f_2 respectively.



Furthermore if one of the focal lengths f_1/f_2 is too short in ist amount, the lens is hard to manufacture. If for this case a limiting value of f_1 > 10 mm and f_2 < -10 mm is assumed, we get the following graph for the range of applicability. It shows, that realistic the telephoto factor in the range k=0.5...1 is useful.