Design and Correction of Optical Systems

Lecture 8: Further performance criteria
2015-06-03
Herbert Gross
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2. Edges and lines
3. Pupil aberrations
4. Sine condition
5. Induced aberrations
6. Vectorial aberrations
7. Fourier imaging formation
8. Caustics
Centroid Ray

- Deviation of centroid ray from chief ray
  - First possibility:
    - asymmetrical apodization
    - coincidence in image plane
  - Second possibility:
    - coma phase aberration
    - coincidence in pupil
Line of Sight

- Chief ray: centroid of geometrical pupil area
  \[ y_{c}^{(CR)}(z) = \frac{\iint y \, dx \, dy}{\int dx \, dy} = \frac{1}{A} \int y \, dx \, dy \]

- Centroid ray: centroid of energy
  \[ y_{c}^{(Cen)}(z) = \frac{\iint y \cdot I(x, y, z) \, dx \, dy}{\iint I(x, y, z) \, dx \, dy} = \frac{1}{P} \int y \cdot I(x, y, z) \, dx \, dy \]

- Due to wave equation the centroid propagates along a straight line: Line of sight

- Wave aberrations of odd order in the azimuthal term influence the centroid
  - tilt and coma-like aberrations of any order
  - centroid has an offset against the peak of intensity
  - simple calculation possible:
  \[ y_{c}^{(CR)}(z) = \frac{2 \cdot z}{A_{ExP}} \cdot \sum_{n=1,3,5,\ldots} \sqrt{2(n + 1)} \cdot c_{n1} \]
- Pupil with apodization
  e.g. non-homogeneous asymmetrical illumination
- Defocus: centroid moves on a straight line (line of sight)
- Peak of intensity moves on a curve (bananicity)
Point Spread Function with Apodization

- Apodisation of the pupil:
  1. Homogeneous
  2. Gaussian
  3. Bessel
- Psf in focus:
  different convergence to zero for larger radii
- Encircled energy:
  same behavior
- Complicated:
  Definition of compactness of the central peak:
  1. FWHM:
     Airy more compact as Gauss
     Bessel more compact as Airy
  2. Energy 95%:
     Gauss more compact as Airy
     Bessel extremly worse
- Small aperture:
  - Diffraction limited
  - Spot size corresponds to Airy diameter
  - Spot size depends on wavelength

- Large aperture:
  - Diffraction negligible
  - Aberration limited
  - Geometrical effects not wavelength dependent
  - But: small influence of dispersion

Log $D_{foc}$

$f = 1000, 500, 200, 100, 50, 20, 10 \text{ mm}$

$\lambda = 10 \mu m$

$\lambda = 1 \mu m$

$\lambda = 550 \text{ nm}$

Log $\sin u$
Depth of Focus: Diffraction Consideration

- Normalized axial intensity for uniform pupil amplitude

\[ I(u) = I_0 \cdot \left( \frac{\sin u}{u} \right)^2 \]

- Decrease of intensity onto 80%:

\[ \frac{1}{2} \Delta z_{\text{diff}} = 0.493 \cdot \frac{\lambda}{n \cdot \sin^2 \theta} \approx \frac{1}{2} \cdot R_u \]

- Scaling measure: Rayleigh length
  - geometrical optical definition
  - depth of focus: \( 1R_E \)

\[ R_u = \frac{\lambda}{n' \sin^2 \theta'} = \frac{n' \cdot \lambda}{NA^2} \]

- Gaussian beams: similar formula

\[ R_u = \frac{\lambda}{n' \pi \theta_o^2} \]
Fresnel Edge Diffraction

- Diffraction at an edge in Fresnel approximation

- Intensity distribution, Fresnel integrals $C(x)$ and $S(x)$

$$I(t) = \frac{1}{2} \left[ \left( \frac{1}{2} - C(t) \right)^2 + \left( \frac{1}{2} - S(t) \right)^2 \right]$$

scaled argument

$$t = \sqrt{\frac{k}{\lambda \cdot \pi}} \cdot x = \sqrt{\frac{2}{\lambda \cdot z}} \cdot x = \sqrt{2N_F}$$

- Intensity:
  - at the geometrical shadow edge: 0.25
  - shadow region: smooth profile
  - bright region: oscillations
- ESF with defocussing

- ESF with spherical aberration

Incoherent Edge Spread Function

\[ I_{ESF}(x) \]

\[ W_{20} = 0.0 \]
\[ W_{20} = 0.1 \]
\[ W_{20} = 0.2 \]
\[ W_{20} = 0.3 \]
\[ W_{20} = 0.4 \]
\[ W_{20} = 0.5 \]
\[ W_{20} = 0.7 \]
- Line image: integral over point spread function
  LSF: line spread function
- Realization: narrow slit convolution of slit width
- But with deconvolution, the PSF can be reconstructed

\[ I_{LSF}(x) = \int I_{PSF}(x, y) dy \]
Line Spread Function

- Line image:
  Fourier transform of pupil in one dimension

- Line spread function with aberrations
  Here: defocussing

\[
I_{LSF}(x_i) = \frac{\int \left| \int P(x_p, y_p) \cdot e^{-\frac{2\pi i}{\lambda R} x_i x_p} \, dx_p \right|^2 \, dy_p}{\int \left[ \int \left| P(x_p, y_p) \right|^2 \, dx_p \right]^2 \, dy_p}
\]
ESF, PSF and ESF-Gradient

- Typical behavior of intensity of an edge image for residual aberrations
- The width of the distribution roughly corresponds to the diameter of the PSF
- Derivative of the edge spread function:
  
  edge position at peak location

![Graph showing ESF, PSF, and derivation of edge spread function](attachment:graph.png)
Pupil Aberrations

- Spherical aberration of the chief ray / pupil imaging
- Exit pupil location depends on the field height
Pupil Aberration

- Interlinked imaging of field and pupil
- Distortion of object imaging corresponds to spherical aberration of the pupil imaging
- Corrected spherical pupil aberration: tangent condition

\[
\frac{\tan w'}{\tan w} = \text{const.}
\]

<table>
<thead>
<tr>
<th>Object imaging</th>
<th>Pupil imaging</th>
</tr>
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<tr>
<td>Blue rays</td>
<td>Marginal rays</td>
</tr>
<tr>
<td>Red rays</td>
<td>Chief rays</td>
</tr>
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</table>
- Eyepiece with pupil aberration

- Illumination for decentered pupil: dark zones due to vignetting
Sine Condition

- Lagrange invariante for paraxial angles U, U'
- sin-condition: extension for finite aperture angle u
- Corresponds to energy conservation in the system
- Constant magnification for all aperture zones
- Pupil shape for finite aperture is a sphere
- Definition of violation of the sine condition:
  OSC (offense against sine condition)
- OSC = 0 means correction of sagittal coma (aplanatic system)

\[ ny \sin U = n' y' \sin U' \]

\[ ny \sin u = n' y' \sin u' \]

\[ m = \frac{nU}{n'U'} = \frac{n \sin u}{n' \sin u'} \]
- Optical path difference for two object points between object and image space
If for example a small field area and a widespread ray bundle is considered, a perfect imaging is possible.

The eikonal with the expression

\[ \delta L = n's' \cdot d\vec{r}' - n\vec{s} \cdot d\vec{r} \]

can be written for \( \delta L = 0 \) as

\[ n \cdot \vec{s} \cdot d\vec{r} = n' \cdot \vec{s}' \cdot d\vec{r}' \]

\[ n \cdot dr \cdot \cos \theta = n' \cdot dr' \cdot \cos \theta' \]

\[ n \cdot \cos \theta = n' \cdot \beta \cdot \cos \theta' \]

In the special case of an angle 90° we get with \( \cos(\theta) = \sin(u) \) the Abbe sine condition

\[ m = \frac{n \sin u}{n' \sin u'} \]

with the lateral magnification

\[ m = \frac{d\vec{r}'}{d\vec{r}} \]
Tangential and Sagittal Coma

- 2 terms of tangential transverse aberration:
  - Sagittal coma depends on $x_p$, describes the asymmetry
  - Tangential coma depends on $y_p$, corresponds to spherical aberration under skew conditions larger by a factor of 3
- Only asymmetry removed with sine condition: sagittal coma vanishes

\[
\Delta y' = -R \cdot (x_p^2 + 3y_p^2) = \Delta y_s' + \Delta y_t'
\]
Decomposition of coma:
1. part symmetrical around chief ray: skew spherical aberration
\[ \Delta y_{\text{skew sph}} = \frac{\Delta y_{\text{upcom}} + \Delta y_{\text{lowcom}}}{2} \]
2. asymmetrical part: tangential coma
\[ \Delta y_{\text{tang coma}} = \frac{\Delta y_{\text{upcom}} - \Delta y_{\text{lowcom}}}{2} \]

Skew spherical aberration:
- higher order aberration
- caustic symmetric around chief ray
Transfer of Energy in Optical Systems

- Conservation of energy
  \[ d^2 P = d^2 P' \]
- Invariant local differential flux
  \[ d^2 P = L \cdot \sin u \cdot \cos u \cdot dA \cdot du \cdot d\phi \]
- Assumption: no absorption
  \[ T = 1 \]
- Delivers the sine condition
  \[ n y \cdot \sin u = n' y' \cdot \sin u' \]
Sine Condition

- Sine condition not fulfilled:
  - nonlinear scaling from entrance to exit pupil
  - spatial filtering on warped grid, nonlinear sampling of spatial frequencies
  - pupil size changes
  - apodization due to distortion
  - wave aberration could be calculated wrong
  - quantitative measure of offence against the sine condition (OSC):
    distortion of exit pupil grid
    \[ D_p = \frac{x_{ap}}{f \cdot n \cdot \sin u} - 1 \]
- Photometric effect of pupil distortion: illumination changes at pupil boundary
- Effect induces apodization
- Sign of distortion determines the effect: outer zone of pupil brighter / darker
- Additional effect: absolute diameter of pupil changes

OSC and Apodization
General Aplanatic Surface

- General approach of Fermat principle: aplanatic surface
  \[ n \cdot \sqrt{r^2 + (z - s)^2} + n' \cdot \sqrt{r^2 + (s' - z)^2} = s'n' - ns \]
- Cartesian oval, 4th order
- Special case \( \text{OPD} = 0 \): \[ s'n' = ns \]

Solution is spherical aplanatic surface

\[
\begin{align*}
n \cdot \sqrt{r^2 + (z - s)^2} + n' \cdot \sqrt{r^2 + (n/n's - z)^2} &= 0 \\
(n/n')^2 \cdot (r^2 + z^2 + s^2 - 2zs) &= (r^2 + z^2 + (n/n')^2 s^2 - 2zs n/n') \\
z^2 (n^2/n'^2 - 1) - 2zs \left[ (n/n')^2 - n/n' \right] + r^2 (n^2/n'^2 - 1) &= 0 \\
\left[ z - \frac{sn}{n + n'} \right]^2 + r^2 &= \left[ \frac{sn}{n + n'} \right]^2
\end{align*}
\]
Isoplanatism Condition of Staebble-Lihotzky

- Sagittal coma aberration: from the geometry of the figure and Lagrange invariant

- Condition of Staebble-Lihotzky

- Problems:
  - no quantitative measure
  - only tangential rays are considered
  - integral criterion

\[
\Delta y'_s = \frac{y'}{m} \left[ \frac{n \sin u}{n' \sin u'} \cdot \frac{S' - s'_p'}{S' - s'_p' + \Delta s_{sph}} - m \right]
\]

\[
s' - s'_p' = \frac{S' - s'_p'}{m} \left( \frac{n \sin u}{n' \sin u'} - m \right)
\]
**Piecewise Isoplanatism**

- Invariance of PSF: to be defined
- Possible options:
  1. relative change of Strehl
  2. correlation of PSF's
- Examples for microscopic lenses with and without flattening correction
- In medium field size: small isoplanatic patches
- On axis: large isoplanatic area
- Criteria not useful at the edge for low performance

<table>
<thead>
<tr>
<th>System</th>
<th>MO plane 100x1.25 isoplanatic patch size in µm</th>
<th>MO not plane 40x0.85 isoplanatic patch size in µm</th>
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<tr>
<td></td>
<td>Strehl 1%</td>
<td>Psf correlation 0.5%</td>
</tr>
<tr>
<td>on axis</td>
<td>70</td>
<td>72</td>
</tr>
<tr>
<td>half field</td>
<td>3.8</td>
<td>3.8</td>
</tr>
<tr>
<td>field zone</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>full field</td>
<td>45</td>
<td>3.8</td>
</tr>
</tbody>
</table>

![Graph showing Strehl and correlation curves for different system conditions]
Conradys OSC (offense against sine condition):
- measurement of deviation of sagittal coma
- quantitative validation of the sine condition

\[
\Delta_{OSC} = \frac{y_t' - y_s'}{y_t'} = 1 - \frac{n \sin u}{m \cdot n' \sin u'} \cdot \frac{S' - s_p'}{s' - s_p'}
\]

Only sagittal coma considered
in case of OSC=0 the Staebke-Lihotzky-condition is automatically fulfilled

\[
W_{coma}\(y, r_p, 0\) = r_p \cdot y_t \cdot \Delta_{OSC}
\]

\[
\Delta y_t' = -3 y \left( m - \frac{n \sin u}{n' \sin u'} \right)
\]

OSC allows for the definition of surface contribution

\[
\Delta_{OSC} = \frac{\sin w_1}{\sin u_1} \sum_{k} (Q_k' - Q_k) \cdot n_k' i^{(CR)}_k
\]
### Overview on conditions for aberrations and aplanatism-isoplanatism

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<th>Nr</th>
<th>Sine cond.</th>
<th>Iso-planat cond.</th>
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<th>Sagittal coma</th>
<th>Tangential coma</th>
<th>Imaging system</th>
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<tr>
<td>1</td>
<td>#</td>
<td>#</td>
<td></td>
<td>#</td>
<td>#</td>
<td>#</td>
<td>general</td>
</tr>
<tr>
<td>2a</td>
<td>#</td>
<td>✓</td>
<td>OSC=0, Conrady</td>
<td>#</td>
<td>0</td>
<td>#</td>
<td>isoplanatic-I</td>
</tr>
<tr>
<td>2b</td>
<td>#</td>
<td>✓</td>
<td>Staebble-Lihotzky / Berek</td>
<td>#</td>
<td>0</td>
<td>0</td>
<td>isoplanatic-II</td>
</tr>
<tr>
<td>3a</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>0</td>
<td>0</td>
<td></td>
<td>axial aplanatic</td>
</tr>
<tr>
<td>3b</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>0 (skew)</td>
<td>0</td>
<td>0</td>
<td>off-axis aplanatic</td>
</tr>
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#### Table for Isoplanatism

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<tr>
<td>Staebble-Lihotzky</td>
<td>Aplanatism</td>
</tr>
</tbody>
</table>

| Tangential coma | 0 | 0 |
| Sagittal coma   | 0 | 0 |
| Spherical aberration | 0 | 0 |
| Skew Spherical aberration | 0 | 0 |
- Special idea of Seidel to consider the 3rd order as a perturbation of the paraxial ray
- Independent changes/contributions of every surface aberration to the final transverse aberration
- Therefore special reference on paraxial fundamental properties
- Aberration expansion: perturbation theory

- Linear independent contributions only in lowest correction order:
  Surface contributions of Seidel additive

- Higher order aberrations (5th order,...): nonlinear superposition
  - 3rd order generates different ray heights and angles at next surfaces
  - induces aberration of 5th order
  - together with intrinsic surface contribution: complete error

- Separation of intrinsic and induced aberrations: refraction at every surface in the system
• Example Gabor telescope
  - a lens pre-corrects a spherical mirror to obtain vanishing spherical aberration
  - due to the strong ray deviation at the plate, the ray heights at the mirror changes significantly
  - as a result, the mirror has induced chromatical aberration, also the intrinsic part is zero by definition

• Surface contributions and chromatic difference (Aldi, all orders)
Vectorial Aberrations

- Wave aberration field
  \[ W(\vec{H}, \vec{r}_p) = \sum_{j,m,n} W_{klm} \cdot (\vec{H} \cdot \vec{H})^j \cdot (\vec{H} \cdot \vec{r}_p)^m \cdot (\vec{r}_p \cdot \vec{r}_p)^n \]
  
  \[ k = 2j + m, \quad l = 2n + m \]

- Normalized field vector: \( \vec{H} \)
  normalized pupil vector: \( \vec{r}_p \)
  angle between \( \vec{H} \) and \( \vec{r}_p \): \( \theta \)

- Expansion according to the invariants
  for circular symmetric components

\[ \vec{H} \cdot \vec{H} = H^2, \quad \vec{r}_p \cdot \vec{r}_p = r_p^2, \quad \vec{H} \cdot \vec{r}_p = H \cdot r_p \cdot \cos \theta \]
## Vectorial Aberrations

- Wave aberration field until the 6th order
- Analogue: transverse aberrations with

$$\Delta \vec{H}' = -\frac{R}{n'} \cdot \nabla_{r_p} W$$

<table>
<thead>
<tr>
<th>ord</th>
<th>j</th>
<th>m</th>
<th>n</th>
<th>Term</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$W_{000}$</td>
<td>uniform Piston</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$W_{200} \cdot (\vec{H} \cdot \vec{H})$</td>
<td>quadratic piston</td>
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<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$W_{111} \cdot (\vec{H} \cdot \vec{r}_p)$</td>
<td>magnification</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$W_{020} \cdot (\vec{r}_p \cdot \vec{r}_p)$</td>
<td>focus</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>$W_{040} \cdot (\vec{r}_p \cdot \vec{r}_p)^2$</td>
<td>spherical aberration</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$W_{131} \cdot (\vec{r}_p \cdot \vec{r}_p) \cdot (\vec{H} \cdot \vec{r}_p)$</td>
<td>coma</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$W_{222} \cdot (\vec{H} \cdot \vec{r}_p)^2$</td>
<td>astigmatism</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$W_{220} \cdot (\vec{H} \cdot \vec{H}) \cdot (\vec{r}_p \cdot \vec{r}_p)$</td>
<td>field curvature</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$W_{311} \cdot (\vec{H} \cdot \vec{H}) \cdot (\vec{H} \cdot \vec{r}_p)$</td>
<td>distortion</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$W_{400} \cdot (\vec{H} \cdot \vec{H})^2$</td>
<td>quartic piston</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>$W_{240} \cdot (\vec{H} \cdot \vec{H}) \cdot (\vec{r}_p \cdot \vec{r}_p)^2$</td>
<td>oblique spherical aberration</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$W_{331} \cdot (\vec{H} \cdot \vec{H}) \cdot (\vec{r}_p \cdot \vec{r}_p) \cdot (\vec{H} \cdot \vec{r}_p)$</td>
<td>coma</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>$W_{422} \cdot (\vec{H} \cdot \vec{H}) \cdot (\vec{H} \cdot \vec{r}_p)^2$</td>
<td>astigmatism</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$W_{420} \cdot (\vec{H} \cdot \vec{H})^2 \cdot (\vec{r}_p \cdot \vec{r}_p)$</td>
<td>field curvature</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>$W_{511} \cdot (\vec{H} \cdot \vec{H})^2 \cdot (\vec{H} \cdot \vec{r}_p)$</td>
<td>distortion</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$W_{600} \cdot (\vec{H} \cdot \vec{H})^3$</td>
<td>piston</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>$W_{060} \cdot (\vec{r}_p \cdot \vec{r}_p)^3$</td>
<td>spherical aberration</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$W_{151} \cdot (\vec{r}_p \cdot \vec{r}_p)^2 \cdot (\vec{H} \cdot \vec{r}_p)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$W_{242} \cdot (\vec{r}_p \cdot \vec{r}_p) \cdot (\vec{H} \cdot \vec{r}_p)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>$W_{333} \cdot (\vec{H} \cdot \vec{r}_p)^3$</td>
<td></td>
</tr>
</tbody>
</table>
Systems with Non-Axisymmetric Geometry

- Wave aberration

\[ W(\bar{H}, \vec{r}_p) = \sum_q \sum_{j,m,n} W_{kln} \cdot [(\bar{H}_0 - \vec{\sigma}_q) \cdot (\bar{H}_0 - \vec{\sigma}_q)]^j \cdot [(\bar{H}_0 - \vec{\sigma}_q) \cdot \vec{r}_p]^m \cdot (\vec{r}_p \cdot \vec{r}_p)^n \]

with shift vector

\[ \bar{H}_j = \bar{H}_{j0} - \vec{\sigma}_j \]

- In 3rd order:
  1. spherical

\[ W(\bar{H}, \vec{r}_p) = \sum_q W_{040.q} \cdot (\vec{r}_p \cdot \vec{r}_p)^2 \]

  + \[ \sum_q W_{131.q} \cdot \bar{H}_0 - \sum_q W_{131.q} \cdot \vec{\sigma}_q \cdot (\vec{r}_p \cdot \vec{r}_p) \cdot \vec{r}_p \]

- 2. coma

\[ + \left[ \frac{1}{2} \left( \sum_q W_{220.q} \right) \cdot \bar{H}_0^2 - \left( \sum_q W_{222.q} \cdot \vec{\sigma}_q \right) \cdot \bar{H}_0 + \frac{1}{2} \left( \sum_q W_{222.q} \cdot \vec{\sigma}_q^2 \right) \cdot \vec{r}_p^2 \right] \]

- 3. astigmatism

\[ + \left[ \sum_q W_{220.q} + \frac{1}{2} \sum_q W_{222.q} \right] \cdot \bar{H}_0^2 - \left[ 2 \sum_q W_{22.q} \cdot \vec{\sigma}_q + \frac{1}{2} \sum_q W_{222.q} \cdot \vec{\sigma}_q^2 \right] \cdot \bar{H}_0 \]

\[ + \left( \sum_q W_{220.q} + \frac{1}{2} \sum_q W_{222.q} \right) \cdot \vec{\sigma}_q^2 \]

- 4. defocus

\[ + \left[ \sum_q W_{311.q} \cdot \bar{H}_0^2 \cdot \bar{H}_0 - \sum_q W_{311,q} \cdot (\bar{H}_0 \cdot \vec{\sigma}_q) \cdot \bar{H}_0 + \sum_q W_{311,q} \cdot \vec{\sigma}_q^2 \cdot \bar{H}_0 \right] \cdot \vec{r}_p \]

\[ + \left[ 2 \sum_q W_{311,q} \cdot (\bar{H}_0 \cdot \vec{\sigma}_q) \cdot \vec{\sigma}_q \right] \cdot \bar{H}_0 \]

- 5. distortion

\[ + \left[ \sum_q W_{311,q} \cdot (\bar{H}_0 \cdot \vec{\sigma}_q) \cdot \vec{\sigma}_q - \sum_q W_{311,q} \cdot \vec{\sigma}_q^2 \cdot \vec{\sigma}_q \right] \cdot \vec{r}_p \]
**Nodal Theory**

- Expanded and rearranged 3rd order expressions:
  - aberrations fields
  - nodal lines/points for vanishing aberration

- Example coma:

\[
W_{\text{coma}} = \left( \sum_q W_{131,q} \right) \cdot \left[ \tilde{H}_o - \sum_q \tilde{\sigma}_q \cdot W_{131,q} \cdot \tilde{r}_p \right] \cdot \left( \tilde{r}_p \cdot \tilde{r}_p \right)
\]

abbreviation: nodal point location

\[
\tilde{a}_{131} = \frac{\sum_q \tilde{\sigma}_q \cdot W_{131q}}{\sum_j W_{131q}} = \frac{\sum_q \tilde{\sigma}_q \cdot W_{131q}}{W_{131}^{(c)}}
\]

one nodal point with vanishing coma

\[
W_{\text{coma}} = W_{131}^{(c)} \cdot \left[ \left( \tilde{H}_o - \tilde{a}_{131} \right) \cdot \tilde{r}_p \right] \cdot \left( \tilde{r}_p \cdot \tilde{r}_p \right)
\]
HMD Projection Lens

- Refractive 3D-system
- Free-formed prism
- One coma nodal point
- Two astigmatism nodal points

\[ W_{\text{rms}}, 0.17 \ldots 0.58 \lambda \]

- Free-formed surface
- Total internal reflection
- Field angle 14°
Fourier Optics – Point Spread Function

- Optical system with magnification $m$
  - Pupil function $P$,
  - Pupil coordinates $x_p, y_p$

  \[
  g_{psf}(x, y, x', y') = N \cdot \iint P(x_p, y_p) \cdot e^{\frac{ik}{z} [x_p \cdot (x' - mx) + y_p \cdot (y' - my)]} \, dx_p \, dy_p
  \]

- PSF is Fourier transform of the pupil function (scaled coordinates)

  \[
  g_{psf}(x, y) = N \cdot \hat{F}[P(x_p, y_p)]
  \]
Fourier Theory of Incoherent Image Formation

- Transfer of an extended object distribution \( I(x,y) \)
- In the case of shift invariance (isoplanasity): incoherent convolution
- Intensities are additive

\[
I_{inc}(x', y') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| g_{psf}(x', x, y', y) \right|^2 \cdot I(x, y) dx dy
\]

\[
I_{inc}(x', y') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| g_{psf}(x'-x, y'-y) \right|^2 \cdot I(x, y) dx dy
\]

\[
I_{image}(x', y') = I_{psf}(x, y) * I_{obj}(x, y)
\]
Fourier Theory of Incoherent Image Formation

- Object intensity $I(x,y)$
- Squared PSF, intensity-response $I_{psf}(x_p,y_p)$
- Image intensity $I'(x',y')$
- Object intensity spectrum $I(v_x,v_y)$
- Optical transfer function $H_{OTF}(v_x,v_y)$
- Image intensity spectrum $I'(v'_x,v'_y)$

Transforms:
- Fourier transform
- Convolution
- Produkt

Results:
- Fourier transform
- Convolution
- Produkt
Testchart Visual Acuity

Snellen test chart
Visual Acuity

- Recognition of simple geometrical shapes:
  1. Landolt ring with gap
  2. Letter 'E'
- Blur of image on retina with distance
Siemens Star Test Plate
Real Image with Different Chromatical Aberrations

original object

good image

color astigmatism 2 λ

6% lateral color

axial color 4 λ
Caustics

- Early investigations on caustics: Leonardo da Vinci 1508
- Caustics at mirrors and lenses

Ref: J. Nye, Natural focusing
Caustics

- More general:
  caustic occurs at every wavefront with concave shape as locus of local curvature

- Physically:
  - crossing of rays indicates a caustic
  - interference with diffraction ripple and ringing is seen

Ref: J. Nye, *Natural focusing*
Ref: W. Singer

![Diagram showing caustics phenomenon](image-url)
Caustics

- Caustic: envelope of rays
- Locus of local curvature
- Calculation:
  
  Caustic: 
  \[ \vec{r}_c = (x_c, y_c, z_c) \]

  Ray direction: 
  \[ \vec{s} = (s_x, s_y, s_z) \]

  Rays: 
  \[ \vec{r}_c = \vec{r} + L \cdot \vec{s} \]

  L distance PC

  Variation of point on wavefront:

  Solution condition for linear system:
  
  Equation of caustic

\[
\begin{vmatrix}
1 + L \frac{\partial s_x}{\partial x} & L \frac{\partial s_x}{\partial y} & s_x \\
L \frac{\partial s_y}{\partial x} & 1 + L \frac{\partial s_y}{\partial y} & s_y \\
s_x & s_y & 1
\end{vmatrix} = 0
\]

\[
x_c = x + L \cdot s_x
\]

\[
y_c = y + L \cdot s_y
\]

\[
z_c = L \cdot s_z
\]

\[
\delta x + L \cdot \frac{\partial s_x}{\partial x} \cdot \delta x + L \cdot \frac{\partial s_x}{\partial y} \cdot \delta y + s_x \cdot \delta L = 0
\]

\[
\delta y + L \cdot \frac{\partial s_y}{\partial x} \cdot \delta x + L \cdot \frac{\partial s_y}{\partial y} \cdot \delta y + s_y \cdot \delta L = 0
\]

\[
L \cdot \frac{\partial s_z}{\partial x} \cdot \delta x + L \cdot \frac{\partial s_z}{\partial y} \cdot \delta y + s_z \cdot \delta L = 0
\]
Caustics

- Special case of one dimension $x$-$z$
- Example: spherical aberration for focussing through plane interface
- Ray direction
  \[ s_x = \frac{\partial W}{\partial x} \]
- Variation
  \[ \left(1 + L \cdot \frac{\partial s_x}{\partial x}\right) \cdot \delta x + s_x \cdot \delta L = 0 \]
  \[ L \cdot \frac{\partial s_z}{\partial x} \cdot \delta x + s_z \cdot \delta L = 0 \]
- Geometry and law of refraction
  \[ s_x = \frac{\partial W}{\partial x} = -n \cdot \frac{x}{a} = -n \cdot \frac{x}{\sqrt{q^2 + x^2}} \]
  \[ L = -\frac{s_x^2}{\partial s_x/\partial x} = a \cdot \frac{1 - (n^2 - 1) \cdot \frac{x^2}{q^2}}{n} \]
- Approximation of small $x$: caustic curve
  \[ z_c = \frac{q}{n} - \frac{3}{2n} \cdot \left[ (n^2 - 1) \cdot q \right]^{1/3} \cdot x_c^{2/3} \]