Design and Correction of optical Systems

Part 4: Paraxial optics

Summer term 2012
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4.1 Imaging - basic notations
- paraxial approximation
- linear collineation
- graphical image construction
- lens makers formula

4.2 Optical system properties - special aspects
- imaging
- multiple components

4.3 Matrix calculus - simple matrices
- relations

4.4 Phase space - basic idea
- invariant
- Optical Image formation:
  All ray emerging from one object point meet in the perfect image point

- Region near axis:
  gaussian imaging
  ideal, paraxial

- Image field size:
  Chief ray

- Aperture/size of light cone:
  marginal ray
  defined by pupil stop
- Single surface between two media
  Radius $r$, refractive indices $n$, $n'$

\[
\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{r} = \frac{1}{f'}
\]

- Imaging condition, paraxial

- Abbe invariant
  alternative representation of the imaging equation

\[
Q_s = n \cdot \left(\frac{1}{r} - \frac{1}{s}\right) = n' \cdot \left(\frac{1}{r} - \frac{1}{s'}\right)
\]
- Law of refraction

\[ n \cdot \sin I = n' \cdot \sin I' \]

- Expansion of the sine-function:

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]

- Linearized approximation of the law of refraction: \( I \longrightarrow i \)

\[ n \cdot i = n' \cdot i' \]

- Relative error of the approximation

\[ \varepsilon = \frac{i' - I'}{I'} = \frac{n \cdot i}{n'} - 1 \]

\[ \arcsin \left( \frac{n \cdot \sin i}{n'} \right) \]
Linear Collineation

- **General rational transformation**

\[
x' = \frac{F_1}{F_0}, \quad y' = \frac{F_2}{F_0}, \quad z' = \frac{F_3}{F_0}
\]

- **Linear expression**

\[
F_j = a_j x + b_j y + c_j z + d_j, \quad j = 0, 1, 2, 3
\]

- **Describes linear collinear transform** 

\[
x = \frac{F_1'}{F_0'}, \quad y = \frac{F_2'}{F_0'}, \quad z = \frac{F_3'}{F_0'}
\]

- **Analog in the image space**

\[
F_j' = a_j' x' + b_j' y' + c_j' z' + d_j', \quad j = 0, 1, 2, 3
\]

- **Inserted in only 2 dimensions**

\[
z' = \frac{c_3 z + d_3}{c_0 z + d_0}, \quad y' = \frac{a_1 y}{c_0 z + d_0}
\]

- **Focal lengths**

\[
f = \frac{a_1}{c_0}, \quad f' = \frac{c_3 d_0 - d_3 c_0}{a_1 c_0}
\]

- **Principal planes**

\[
z_p = \frac{a_1 - d_0}{c_0}, \quad z_p' = \frac{c_3 a_1 - c_3 d_0 + d_3 c_0}{a_1 c_0}
\]
- Special choice of origin of coordinate systems: Newton imaging equations

- Finite angles: $\tan(u)$ must be taken:
  Magnification:
  
  $$m = \frac{\tan u'}{\tan u}$$

  Focal length:
  
  $$\frac{1}{f'} = \frac{\tan u' - \tan u}{h}$$

  Invariant:
  
  $$ny \tan u = n' y' \tan u'$$
- Positive lens
  Real image for $s > f$

- Negative lens
  Virtual image
Graphical image construction according to Listing by 3 special rays:

1. First parallel through axis, through focal point in image space $F'$

2. First through focal point $F$, then parallel to optical axis

3. Through nodal points, leaves the lens with the same angle

Procedure work for positive and negative lenses
For negative lenses the $F / F'$ sequence is reversed
First ray parallel to arbitrary ray through focal point, becomes parallel to optical axis

Arbitrary ray:
- constant height in principal planes $S \rightarrow S'$
- meets the first ray in the back focal plane, desired ray is $S'Q$
- Principle setups of the image of a positive lens
- Object and image can be real/virtual
- Ranges of imaging
  Location of the image for a single lens system

- Change of object location

- Image could be:
  1. real / virtual
  2. enlarged/reduced
  3. in finite/infinite distance
- Lateral magnification for finite imaging
- Scaling of image size

\[ m = \frac{y'}{y} = -\frac{f \cdot \tan u}{f' \cdot \tan u'} \]
- Imaging on axis: circular / rotational symmetry
  Only spherical aberration and chromatical aberrations

- Finite field size, object point off-axis:
  - chief ray as reference
  - skew ray bundles: coma and distortion
  - Vignetting, cone of ray bundle not circular symmetric
  - to distinguish: tangential and sagittal plane
Afocal systems with object/image in infinity
Definition with field angle $w$
angular amplification

\[ \gamma = \frac{\tan w'}{\tan w} = \frac{nh}{n' h'} \]

Relation with finite-distance magnification

\[ \beta \cdot \gamma = -\frac{f}{f'} \]
Axial magnification

Approximation for small $\Delta z$ and $n = n'$

\[
\alpha = \frac{\Delta z'}{\Delta z} = -\beta^2 \cdot \frac{f'}{f} \cdot \frac{1}{1 - \frac{\beta \cdot \Delta z}{f}}
\]

\[
\alpha = -\beta^2 = -\frac{\tan^2 u}{\tan^2 u'}
\]
- Distance object-image: (transfer length)
  \[ L = f' \left( 2 + m + \frac{1}{m} \right) \]

- Two solution for a given L with different magnifications
  \[ m = \frac{L}{2f'} - 1 \pm \sqrt{\left( \frac{L}{2f'} \right)^2 - \frac{L}{f'}} \]

- No real imaging for \( L < 4f \)
Magnification parameter $M$:
defines ray path through the lens

\[ M = \frac{U' + U}{U' - U} = \frac{1 + \beta}{1 - \beta} = \frac{2f}{s} + 1 = \frac{2f}{s'} - 1 \]

Special cases:
1. $M = 0$: symmetrical 4f-imaging setup
2. $M = -1$: object in front focal plane
3. $M = +1$: object in infinity

The parameter $M$ strongly influences the aberrations
- **Image in infinity:**
  - collimated exit ray bundle
  - realized in binoculars

- **Object in infinity**
  - input ray bundle collimated
  - realized in telescopes
  - aperture defined by diameter not by angle
Imaging by a lens in air:

- **Lens Maker's Formula**
  \[
  \frac{1}{s'} - \frac{1}{s} = \frac{1}{f}
  \]

- **Magnification**
  \[
  \beta = \frac{s'}{s}
  \]

- **Real imaging:**
  \( s < 0, \; s' > 0 \)

- **Intersection lengths** \( s, s' \) measured with respective to the principal planes \( P, P' \)
Imaging equation in inverse representation with refractive power and vergence $1/s$: linear behavior
Imaging equation according to Newton:

\[ z \cdot z' = f \cdot f' \]

distances \( z, z' \) measured relative to the focal points
- Two lenses with distance $d$
  
  \[ F = F_1 + F_2 - \frac{d \cdot F_1 \cdot F_2}{n} \]

- Focal length
distance of inner focal points $e$
  
  \[ f = \frac{f_1 \cdot f_2}{f_1 + f_2 - d} = \frac{f_1 \cdot f_2}{e} \]

- Sequence of thin lenses close together
  
  \[ F = \sum_{k} F_k \]

- Sequence of surfaces with relative ray heights $h_j$, paraxial
  
  \[ F = \sum_{k} \frac{h_k}{h_1} \cdot (n'_k - n_k) \cdot \frac{1}{r_k} \]

- Magnification
  
  \[ \beta = \frac{s'_1}{s_1} \cdot \frac{s'_2}{s_2} \cdots \frac{s'_k}{s_k} \cdot \frac{n_1}{n'_k} \]
- **Focal length**
  - \( e \): tube length

- **Image location**

\[
f' = \frac{f'_1 \cdot f'_2}{f'_1 + f'_2 - d} = \frac{f'_1 \cdot f'_2}{e}
\]

\[
s'_2 = \frac{(f'_1 - d) \cdot f'_2}{f'_1 + f'_2 - d} = \frac{(f'_1 - d) \cdot f'}{f'_1}
\]
- Paraxial raytrace transfer
  \[ y_j = y_{j-1} + d_{j-1} \cdot U_{j-1} \quad U_j' = U_{j-1} \]

- Matrix formulation
  \[
  \begin{pmatrix}
  y_j' \\
  U_j'
  \end{pmatrix} =
  \begin{pmatrix}
  1 & d_{j-1} \\
  0 & 1
  \end{pmatrix}
  \begin{pmatrix}
  y_j \\
  U_j
  \end{pmatrix}
  \]

- Paraxial raytrace refraction
  \[ y_j = y_{j-1} \quad i_j = \rho_j \cdot y_j + U_{j-1} \quad i_j' = \frac{n_j}{n_j'} i_j \]
  \[ U_j' = U_{j-1} - i_j + i_j' \]

- Inserted
  \[ U_j' = \frac{\rho_j \cdot (n_j' - n_j)}{n_j} y_j + \frac{n_j}{n_j'} U_{j-1} \]

- Matrix formulation
  \[
  \begin{pmatrix}
  y_j' \\
  U_j'
  \end{pmatrix} =
  \begin{pmatrix}
  \frac{1}{n_j} & \frac{0}{n_j'} \\
  \frac{-\rho_j \cdot (n_j' - n_j)}{n_j} & \frac{n_j}{n_j'}
  \end{pmatrix}
  \begin{pmatrix}
  y_j \\
  U_j
  \end{pmatrix}
  \]
Matrix formalism for finite angles

\[
\begin{pmatrix}
  y'_j \\
  \tan u'_j
\end{pmatrix} =
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\cdot
\begin{pmatrix}
  y_j \\
  \tan u_j
\end{pmatrix}
\]
- Linear relation of ray transport
- Simple case: free space propagation

- General case: paraxial segment with matrix ABCD-matrix:

\[
\begin{pmatrix} x' \\ u' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = M \begin{pmatrix} x \\ u \end{pmatrix}
\]
Linear transfer of spation coordinate $x$ and angle $u$

- Matrix representation

- Lateral magnification for $u=0$

- Angle magnification of conjugated planes

- Refractive power for $u=0$

- Composition of systems

- Determinant, only 3 variables

\[
\begin{align*}
    x' &= Ax + Bu \\
    u' &= Cx + Du \\

    \begin{pmatrix}
    x' \\
    u'
    \end{pmatrix} &= \begin{pmatrix}
    A & B \\
    C & D
    \end{pmatrix}
    \begin{pmatrix}
    x \\
    u
    \end{pmatrix}
    = M
    \begin{pmatrix}
    x \\
    u
    \end{pmatrix}
    \\

    A &= x'/x = \beta \\
    D &= u'/u = \gamma \\
    C &= u'/x \\

    M &= M_k \cdot M_{k-1} \cdots M_2 \cdot M_1 \\

    \det M &= AD - BC = \frac{n}{n'}
\end{align*}
\]
- System inversion
  \[ M^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \]

- Transition over distance L
  \[ M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \]

- Thin lens with focal length f
  \[ M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \]

- Dielectric plane interface
  \[ M = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{pmatrix} \]

- Afocal telescope
  \[ M = \begin{pmatrix} 1 & L \\ \Gamma & \Gamma \end{pmatrix} \]
Calculation of intersection length

\[ s' = \frac{A \cdot s + B}{C \cdot s + D} \]

Magnifications:
1. lateral

\[ \beta = \frac{AD - BC}{C \cdot s + D} \]

2. angle

\[ \gamma = C \cdot s + D = \frac{AD - BC}{A - C \cdot s'} \]

3. axial, depth

\[ \alpha = \frac{ds'}{ds} = \frac{AD - BC}{(C \cdot s + D)^2} \]

Principal planes

\[ a_H = \frac{AD - BC - D}{C} \]

\[ a_H' = \frac{A - 1}{C} \]

\[ a_F = \frac{A}{C} \]

\[ a_F' = \frac{D}{C} \]
Direct phase space representation of raytrace: spatial coordinate vs angle
Direct phase space representation of raytrace: spatial coordinate vs angle
Grin lens with aberrations in phase space:
- continuous bended curves
- aberrations seen as nonlinear angle or spatial deviations
- Transition pupil-image plane: $90^\circ$ rotation in phase space
- Planes Fourier inverse
- Marginal ray: space coordinate $x$ ---> angle $\theta'$
- Chief ray: angle $\theta$ ---> space coordinate $x'$
- Product of field size $y$ and numerical aperture is invariant in a paraxial system

$$L = n \cdot y \cdot u = n' \cdot y' \cdot u'$$

- The invariant $L$ describes the phase space volume (area)

- The invariance corresponds to
  1. Energy conservation
  2. Liouville theorem
  3. Constant transfer of information
- Geometrical optic: 
  Etendue, light gathering capacity
- Paraxial optic: invariant of Lagrange / Helmholtz
- Invariance corresponds to conservation of energy
- Interpretation in phase space: 
  constant area, only shape is changed at the transfer through an optical system

\[ L_{\text{Geo}} = \frac{D}{2} \cdot \sin u \]
\[ L = n \cdot y \cdot u = n' \cdot y' \cdot u' \]
- Laser optics: beam parameter product
  waist radius times far field divergence angle

- Minimum value of L:
  TEM$_{oo}$ - fundamental mode

- Elementary area of phase space:
  Uncertainty relation in optics

- Laser modes: discrete structure of phase space

- Geometrical optics: quasi continuum

- L is a measure of quality of a beam
  small L corresponds to a good focussability

$L_{GB} = w_o \cdot \theta_o$

$L_{GB} = \frac{\lambda}{\pi}$

$L_{GB} = w_n \cdot \theta_n = \frac{\lambda}{\pi} \cdot (2n+1)$
- Nonlinearity of the law of refraction defines the paraxial approximation
- Linear collineation: general approach of linear mapping, in case of larger angles with \( \tan(u) \)
- Graphical image construction in paraxial optics:
  - 3 rays determine location and size of the image: nodal ray, rays through focal points \( F, F' \)
  - are parallel to axis
- Imaging condition of a single lens: real image for object distances \( s>2f \), virtual images for
  closer object points
- Definition of lateral magnification, angle and depth magnification
- Lens makers formula \(-1/s + 1/s' = 1/f\) allows for paraxial imaging calculations
- Combination of several lenses, cascaded systems
  - For thin components near together: focal power \( F=1/f \) is additive
- Matrix calculus: practical calculation scheme with 2x2 ABCD matrices, connect ray
  coordinate and angle between two planes
- Phase space in optics: spatial coordinate \( y \) and angle \( u \)
- Illustration of optical systems and ray paths by a line in the phase space
- Lagrange invariant: constant area of a ray bundle in phase space, corresponds to the
  conservation of energy
Next lecture: Part 5 – Properties of optical systems
Date: Wednesday, 2012-05-16

Content:
5.1 Pupil
- basic notations
- pupil
- special rays
- vignetting
- ray sets

5.2 Special imaging setups
- telecentricity
- anamorphic imaging
- Scheimpflug condition

5.3 Canonical coordinates
- normalized properties
- pupil sphere

5.4 Delano diagram
- basic idea
- examples