Design and Correction of Optical Systems

Lecture 6: Wave aberrations
2019-05-20
Herbert Gross
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<th>Week</th>
<th>Date</th>
<th>Topic</th>
<th>Details</th>
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<tbody>
<tr>
<td>1</td>
<td>08.04</td>
<td>Basics</td>
<td>Law of refraction, Fresnel formulas, optical system model, raytrace, calculation approaches</td>
</tr>
<tr>
<td>2</td>
<td>15.04</td>
<td>Materials and Components</td>
<td>Dispersion, anormal dispersion, glass map, liquids and plastics, lenses, mirrors, aspheres, diffractive elements</td>
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<tr>
<td>3</td>
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<td>Paraxial Optics</td>
<td>Paraxial approximation, basic notations, imaging equation, multi-component systems, matrix calculation, Lagrange invariant, phase space visualization</td>
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<td>Pupil, ray sets and sampling, aperture and vignetting, telecentricity, symmetry, photometry</td>
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<td>Longitudinal and transverse aberrations, spot diagram, polynomial expansion, primary aberrations, chromatical aberrations, Seidels surface contributions</td>
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<td>6</td>
<td>20.05</td>
<td>Wave Aberrations</td>
<td>Fermat principle and Eikonal, wave aberrations, expansion and higher orders, Zernike polynomials, measurement of system quality</td>
</tr>
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<td>7</td>
<td>27.05</td>
<td>PSF and Transfer function</td>
<td>Diffraction, point spread function, PSF with aberrations, optical transfer function, Fourier imaging model</td>
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<tr>
<td>8</td>
<td>03.06</td>
<td>Further Performance Criteria</td>
<td>Rayleigh and Marechal criteria, Strehl definition, 2-point resolution, MTF-based criteria, further options</td>
</tr>
<tr>
<td>9</td>
<td>17.06</td>
<td>Optimization and Correction</td>
<td>Principles of optimization, initial setups, constraints, sensitivity, optimization of optical systems, global approaches</td>
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<td>10</td>
<td>24.06</td>
<td>Correction Principles I</td>
<td>Symmetry, lens bending, lens splitting, special options for spherical aberration, astigmatism, coma and distortion, aspheres</td>
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<tr>
<td>11</td>
<td>01.07</td>
<td>Correction Principles II</td>
<td>Field flattening and Petzval theorem, chromatical correction, achromate, apochromate, sensitivity analysis, diffractive elements</td>
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<td>12</td>
<td>08.07</td>
<td>Optical System Classification</td>
<td>Overview, photographic lenses, microscopic objectives, lithographic systems, eyepieces, scan systems, telescopes, endoscopes</td>
</tr>
</tbody>
</table>
1. Rays and wavefronts
2. Wave aberrations
3. Expansion of the wave aberrations
4. Zernike polynomials
5. Performance criteria
6. Non-circular pupil areas
7. Measurement of wave aberrations
- Rays and waves carry the same information
- Wave surface is perpendicular on the rays
- Wave is purely geometrical and has no diffraction properties
Law of Malus-Dupin

- Law of Malus-Dupin:
  - equivalence of rays and wavefronts
  - both are orthonormal
  - identical information

- Condition:
  No caustic of rays

- Mathematical:
  Rotation of Eikonal vanish
  \[ \text{rot}(n \cdot \vec{s}) = 0 \]

- Optical system:
  Rays and spherical waves orthonormal
Fermat Principle

- Fermat principle: the light takes the ray path, which corresponds to the shortest time of arrival

- The realized path is a minimum and therefore the first derivatives vanish

\[ \delta L = \delta \int_{P_1}^{P_2} n(x, y, z) \, ds = 0 \]

- Several realized ray pathes have the same optical path length

\[ L = \int_{P_1}^{P_2} n \cdot \vec{s} \cdot d\vec{r} = \text{const.} \]

- The principle is valid for smooth and discrete index distributions
Relationships

- Concrete calculation of wave aberration: addition of discrete optical path lengths (OPL)

- Reference on chief ray and reference sphere (optical path difference)

- Relation to transverse aberrations

- Conversion between longitudinal transverse and wave aberrations

- Scaling of the phase / wave aberration:
  1. Phase angle in radiant
  2. Light path (OPL) in mm
  3. Light path scaled in \( \lambda \)

\[
\begin{align*}
\Delta_{OPD}(x,y) &= l_{OPL}(x,y) - l_{OPL}(0,0) \\
\frac{\partial W}{\partial y_p} &= - \frac{\Delta y'}{R-W} \approx - \frac{\Delta y'}{R} \\
\Delta s' &= \frac{R}{y_p} \cdot \Delta y' = \frac{\Delta y'}{\sin u'} = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p} \\
E(x) &= A(x) \cdot e^{i \cdot \varphi(x)} \\
E(x) &= A(x) \cdot e^{i \cdot k \Delta_{OPD}(x)} \\
E(x) &= A(x) \cdot e^{2 \pi i \cdot W(x)}
\end{align*}
\]
Relationship to Transverse Aberration

- Relation between wave and transverse aberration
- Approximation for small aberrations and small aperture angles $u$
- Ideal wavefront, reference sphere: $W_{\text{ideal}}$
- Real wavefront: $W_{\text{real}}$
- Finite difference

$$\Delta W = W = W_{\text{real}} - W_{\text{ideal}}$$

$$\varphi \approx \tan \varphi = \frac{\partial W}{\partial y_p}$$

$$\Delta y' = - R \cdot \varphi$$

$$\frac{\partial W}{\partial y_p} = - \frac{\Delta y'}{R - W} \approx - \frac{\Delta y'}{R}$$

*Diagram showing ideal and real wavefronts, reference sphere, and rays.*
Wave Aberration in Optical Systems

- Definition of optical path length in an optical system:
  Reference sphere around the ideal object point through the center of the pupil
- Chief ray serves as reference
  Difference of OPL : optical path difference OPD
- Practical calculation: discrete sampling of the pupil area, real wave surface represented as matrix

![Diagram of wave aberration in optical systems]

- Object plane (Op)
- Entrance pupil (EnP)
- Exit plane (ExP)
- Image plane (Ip)
- Upper coma ray
- Lower coma ray
- Chief ray
- Wave aberration
- Wave front
- Reference sphere
Pupil Sampling

- All rays start in one point in the object plane
- The entrance pupil is sampled equidistant
- In the exit pupil, the transferred grid may be distorted
- In the image plane a spreaded spot diagram is generated
Wave Aberrations

- Quality assessment:
  - peak valley value (PV)
  - rms value, area average
  - Zernike decomposition for detailed analysis
Wave Aberrations – Sign and Reference

- Wave aberration: relative to reference sphere
- Choice of offset value: vanishing mean
- Sign of $W$:
  - $W > 0$: stronger convergence; intersection: $s < 0$
  - $W < 0$: stronger divergence; intersection: $s < 0$

\[
\langle W(x, y) \rangle = \frac{1}{F_{Exp}} \iint W(x, y) \, dx \, dy = 0
\]
Tilt of Wavefront

- Change of reference sphere:
  - tilt by angle $\theta$
  - linear in $y_p$
  \[ \Delta W_{\text{tilt}} = n \cdot y_p \cdot \theta \]

- Wave aberration due to transverse aberration $\Delta y'$
  \[ \Delta W_{\text{tilt}} = -\frac{y_p}{R_{\text{Ref}}} \cdot \Delta y' \]

- Is the usual description of distortion
Defocussing of Wavefront

Paraxial defocussing by $\Delta z$:
Change of wavefront

$$\Delta W_{Def} = -\frac{n \cdot r_p^2}{2R_{ref}^2} \cdot \Delta z' = -\frac{1}{2} n \cdot \Delta z' \cdot \sin^2 u$$
Special Cases of Wave Aberrations

- Wave aberrations are usually given as reduced aberrations:
  - wave front for only 1 field point
  - field dependence represented by discrete cases

- Special case of aberrations:
  1. axial color and field curvature:
     represented as defocussing term, Zernike $c_4$

  2. distortion and lateral color:
     represented as tilt term, Zernike $c_2, c_3$
3. afocal system
   - exit pupil in infinity
   - plane wave as reference

4. telecentric system
   chief ray parallel to axis
Expansion of the Wave Aberration

- Table as function of field and aperture
- Selection rules: checkerboard filling of the matrix

<table>
<thead>
<tr>
<th>Field y</th>
<th>Spherical</th>
<th>Coma</th>
<th>Astigmatism</th>
</tr>
</thead>
<tbody>
<tr>
<td>y^0</td>
<td>y^1</td>
<td>y^2</td>
<td>y^3</td>
</tr>
<tr>
<td>Distortion</td>
<td>r^1</td>
<td></td>
<td>y r cosθ</td>
</tr>
<tr>
<td></td>
<td>r^2</td>
<td></td>
<td>Tilt</td>
</tr>
<tr>
<td>Defocus</td>
<td>r^2</td>
<td></td>
<td>y^2 r^2 cos^2θ</td>
</tr>
<tr>
<td>Aper-</td>
<td>r^3</td>
<td></td>
<td>y^3 r^3 cosθ</td>
</tr>
<tr>
<td>ture r</td>
<td>r^4</td>
<td></td>
<td>y^3 r^3 cos^3θ</td>
</tr>
<tr>
<td>Spherical</td>
<td>r^4</td>
<td></td>
<td>y^2 r^4 cos^2θ</td>
</tr>
<tr>
<td>primary</td>
<td>r^5</td>
<td></td>
<td>y^5 r cosθ</td>
</tr>
<tr>
<td>Coma</td>
<td>r^5</td>
<td></td>
<td>Distortion</td>
</tr>
<tr>
<td>y^5</td>
<td>r^6</td>
<td></td>
<td>primary</td>
</tr>
<tr>
<td>secondary</td>
<td>r^6</td>
<td></td>
<td>secondary</td>
</tr>
</tbody>
</table>

Image location

Primary aberrations / Seidel

Secondary aberrations
### Polynomial Expansion of Wave Aberrations

- **Taylor expansion of the wavefront:**
  - $y'$: Image height, index $k$
  - $r_p$: Pupil height, index $l$
  - $\theta$: Pupil azimuth angle, index $m$

- **Symmetry invariance:**
  1. Image height
  2. Pupil height
  3. Scalar product between image and pupil vector

- **Number of terms**
  - sum of indices in the exponent $i_{\text{sum}}$

- **Polynomial Expansion of Wave Aberrations**

  $$W(y', r_p, \theta) = \sum_{k,l,m} W_{klm} y'^k r_p^l \cos^m \theta$$

  - $\vec{y}'$
  - $|\vec{r}_p|$
  - $\vec{y}' \cdot \vec{r}_p = y' \cdot r_p \cdot \cos \theta$

<table>
<thead>
<tr>
<th>$i_{\text{sum}}$</th>
<th>$N_i$ number of terms</th>
<th>Type of aberration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>image location</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>primary aberrations, 4th order</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>secondary aberrations, 6th order</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>8th order</td>
</tr>
</tbody>
</table>
Taylor Expansion of the Primary Aberrations

- Expansion of the monochromatic aberrations
- First real aberration: primary aberrations, 4\textsuperscript{th} order as wave deviation

\[ W(y', r_p, y_p) = A_s r_p^4 + A_c y' r_p^2 y_p + A_a y'^2 y_p^2 + A_p y'^2 r_p^2 + A_d y'^3 y_p \]

- Coefficients of the primary aberrations:
  \( A_S \): Spherical Aberration
  \( A_C \): Coma
  \( A_A \): Astigmatism
  \( A_P \): Petzval curvature
  \( A_D \): Distortion

- Alternatively: expansion in polar coordinates:
  Zernike basis expansion, usually only for one field point, orthogonalized
Zernike Polynomials

- Expansion of the wave aberration on a circular area
  \[ W(r, \varphi) = \sum_{n} \sum_{m=-n}^{n} c_{nm} Z_n^m (r, \varphi) \]

- Zernike polynomials in cylindrical coordinates:
  Radial function \( R(r) \), index \( n \)
  Azimuthal function \( \varphi \), index \( m \)

- Orthonormality

- Advantages:
  1. Minimal properties due to \( W_{\text{rms}} \)
  2. Decoupling, fast computation
  3. Direct relation to primary aberrations for low orders

- Problems:
  1. Computation on discrete grids
  2. Non circular pupils
  3. Different conventions concerning indices, scaling, coordinate system

\[ c_{nm} = \frac{2(n+1)}{\pi(1+\delta_{m0})} \int_0^{2\pi} \int_0^1 W(r, \varphi) Z^*_n (r, \varphi) d\varphi dr \]

\[ Z_n^m (r, \varphi) = R_n^m (r) \begin{cases} 
\sin m\varphi & \text{für } m > 0 \\
\cos m\varphi & \text{für } m < 0 \\
1 & \text{für } m = 0
\end{cases} \]

\[ \int_0^{2\pi} \int_0^1 Z_n^m (r, \varphi) Z^*_n (r, \varphi) d\varphi dr = \frac{\pi \cdot (1+\delta_{m0}) \cdot \delta_{nn'} \delta_{mm'}}{2(n+1) \cdot \delta_{nn'} \delta_{mm'}} \]
Zernike Polynomials

- Expansion of wave aberration surface into elementary functions / shapes

\[ W(r, \phi) = \sum_{n} \sum_{m=-n}^{n} c_{nm} Z_n^m(r, \phi) \]

- Zernike functions are defined in circular coordinates \( r, \phi \)

\[ Z_n^m(r, \phi) = R_n^m(r) \cdot \begin{cases} 
\sin(m\phi) & \text{for } m > 0 \\
\cos(m\phi) & \text{for } m < 0 \\
1 & \text{for } m = 0 
\end{cases} \]

- Ordering of the Zernike polynomials by indices:
  - \( n \) : radial
  - \( m \) : azimuthal, \( \sin/\cos \)

- Mathematically orthonormal function on unit circle for a constant weighting function

- Direct relation to primary aberration types
Azimuthal Dependence of Zernike Polynomials

- Azimuthal spatial frequency
Zernike Polynomials: Meaning of Lower Orders

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>Polar coordinates</th>
<th>Cartesian coordinates</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>piston</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$r \sin \varphi$</td>
<td>$x$</td>
<td>tilt in x</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$r \cos \varphi$</td>
<td>$y$</td>
<td>tilt in y</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>$r^2 \sin 2\varphi$</td>
<td>$2xy$</td>
<td>Astigmatism 45°</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$2r^2 - 1$</td>
<td>$2x^2 + 2y^2 - 1$</td>
<td>defocussing</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$r^2 \cos 2\varphi$</td>
<td>$y^2 - x^2$</td>
<td>Astigmatism 0°</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>$r^3 \sin 3\varphi$</td>
<td>$3xy^2 - x^3$</td>
<td>trefoil 30°</td>
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<tr>
<td>3</td>
<td>-1</td>
<td>$(3r^3 - 2r)\sin \varphi$</td>
<td>$3x^3 - 2x + 3xy^2$</td>
<td>coma x</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>$(3r^3 - 2r)\cos \varphi$</td>
<td>$3y^3 - 2y + 3x^2 y$</td>
<td>coma y</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$r^3 \cos 3\varphi$</td>
<td>$y^3 - 3x^2 y$</td>
<td>trefoil 0°</td>
</tr>
<tr>
<td>4</td>
<td>-4</td>
<td>$r^4 \sin 4\varphi$</td>
<td>$4xy^3 - 4x^3 y$</td>
<td>Four sheet 22.5°</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
<td>$(4r^4 - 3r^2)\sin 2\varphi$</td>
<td>$8xy^3 + 8x^3 y - 6xy$</td>
<td>Secondary astigmatism</td>
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<tr>
<td>4</td>
<td>0</td>
<td>$6r^4 - 6r^2 + 1$</td>
<td>$6x^4 + 6y^4 + 12x^2 y^2 - 6x^2 - 6y^2 + 1$</td>
<td>Spherical aberration</td>
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<tr>
<td>4</td>
<td>2</td>
<td>$(4r^4 - 3r^2)\cos 2\varphi$</td>
<td>$4y^4 - 4x^4 + 3x^2 - 3y^2 - 4x^2 y^2$</td>
<td>Secondary astigmatism</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>$r^4 \cos 4\varphi$</td>
<td>$y^4 + x^4 - 6x^2 y^2$</td>
<td>Four sheet 0°</td>
</tr>
</tbody>
</table>
Zernike Fringe vs Zernike Standard Polynomials

Fringe coefficients:
- Z 4,9,16 = n² spherical
- azimutal order \( \phi \) grows if number increases

Standard coefficients - different term numbers
- \( Z_n = 0.05 \)
- \( \text{RMS} = 0.05\lambda \)
- \( \text{SR} \sim 1 - 40 Z_n^2 = 0.9 \)

Elimination of tilt
- No Elimination of defocus @ Zemax

In radial symmetric system for y-field (meridional) sinus-terms vanish
Balance of Lower Orders by Zernike Polynomials

- Mixing of lower orders to get the minimal $W_{\text{rms}}$

- Example spherical aberration:
  1. Spherical 4th order according to Seidel
  2. Additional quadratic expression: Optimal defocussing for edge correction
  3. Additional absolute term Minimale value of $W_{\text{rms}}$

- Special case of coma: Balancing by tilt contribution, corresponds to shift between peak and centroid

\[
W(r_p) = 6r_p^4 - 6r_p^2 + 1
\]
Zernike Polynomials

- Advantages of the Zernike polynomials:
  - de-coupling due to orthogonality
  - stable numerical computation
  - direct relation of lower orders to classical aberrations
  - optimale balancing of lower orders (e.g. best defocus for spherical aberration)
  - fast calculation of $W_{\text{rms}}$ and Strehl ratio in approximation of Marechal

- Necessary requirements for orthogonality:
  - pupil shape circular
  - uniform illumination of pupil (corresponds to constant weighting)
  - no discretization effects (finite number of points, boundary)

- Different standardizations used concerning indexing, scaling, sign of coordinates (orientation for off-axis field points):
  - Fringe representation: peak value 1, normalized
  - Standard representation $W_{\text{rms}}$ normalized
  - Original representation according to Nijbor-Zernike
  Norm ISO 10110 allows Fringe and Standard representation
Calculation of Zernike Polynomials

- Assumptions:
  1. Pupil circular
  2. Illumination homogeneous
  3. Negligible discretization effects /sampling, boundary)

- Direct computation by double integral:
  1. Time consuming
  2. Errors due to discrete boundary shape
  3. Wrong for non circular areas
  4. Independent coefficients

- LSQ-fit computation:
  1. Fast, all coefficients $c_j$ simultaneously
  2. Better total approximation
  3. Non stable for different numbers of coefficients, if number too low
  4. Stable for non circular shape of pupil

\[ c_j = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} W(r, \varphi) Z_j(r, \varphi) \, d\varphi \, dr \]

\[ \sum_{i=1}^N \left[ W_i - \sum_{j=1}^N c_j Z_j(r_i) \right]^2 = \min \]

\[ \vec{c} = \left( Z^T Z \right)^{-1} Z^T \vec{W} \]
Performance Description by Zernike Expansion

- Vector of $c_j$
  linear sequence with running index

- Sorting by symmetry
Zernike polynomials: Different Conventions

- Different standardizations used concerning:
  1. indexing
  2. scaling / normalization
  3. sign of coordinates (orientation for off-axis field points)

- Fringe - representation
  1. CodeV, Zemax, interferometric test of surfaces
  2. Standardization of the boundary to ±1
  3. no additional prefactors in the polynomial
  4. Indexing according to m (Azimuth), quadratic number terms have circular symmetry
  5. coordinate system invariant in azimuth

- Standard - representation
  - CodeV, Zemax, Born / Wolf
  - Standardization of rms-value on ±1 (with prefactors), easy to calculate Strehl ratio
  - coordinate system invariant in azimuth

- Original - Nijboer - representation
  - Expansion:
    \[ W(r, \varphi) = a_{00} + \frac{1}{\sqrt{2}} \sum_{n=0}^{k} a_{0n} R_n^0 + \sum_{n=0}^{k} \sum_{m=1}^{n} a_{nm} R_n^m \cos(m\varphi) + \sum_{n=0}^{k} \sum_{m=1}^{n} b_{nm} R_n^m \sin(m\varphi) \]
  - Standardization of rms-value on ±1
  - coordinate system rotates in azimuth according to field point
Wave Aberration Criteria

- Mean quadratic wave deviation (\( W_{\text{Rms}} \), root mean square)

\[
W_{\text{rms}} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \frac{1}{A_{\text{Exp}}} \iint [W(x_p, y_p) - W_{\text{mean}}(x_p, y_p)]^2 \, dx_p \, dy_p
\]

with pupil area

\[
A_{\text{Exp}} = \iint dx \, dy
\]

- Peak valley value \( W_{pv} \): largest difference

\[
W_{pv} = \max [W_{\text{max}}(x_p, y_p) - W_{\text{min}}(x_p, y_p)]
\]

- General case with apodization: weighting of local phase errors with intensity, relevance for psf formation

\[
W_{\text{rms}} = \sqrt{\frac{1}{A_{\text{Exp}}^{(w)}} \iint I_{\text{Exp}}(x_p, y_p) \cdot [W(x_p, y_p) - W_{\text{mean}}^{(w)}(x_p, y_p)]^2 \, dx_p \, dy_p}
\]
Rayleigh Criterion

- The Rayleigh criterion \( |W_{PV}| \leq \frac{\lambda}{4} \)

  gives individual maximum aberrations coefficients, depends on the form of the wave

- Examples:

<table>
<thead>
<tr>
<th>aberration type</th>
<th>coefficient</th>
<th>coefficient type</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus</td>
<td>Seidel</td>
<td>( a_{20} = 0.25 )</td>
</tr>
<tr>
<td>defocus</td>
<td>Zernike</td>
<td>( c_{20} = 0.125 )</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Seidel</td>
<td>( a_{40} = 0.25 )</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Zernike</td>
<td>( c_{40} = 0.167 )</td>
</tr>
<tr>
<td>astigmatism</td>
<td>Seidel</td>
<td>( a_{22} = 0.25 )</td>
</tr>
<tr>
<td>astigmatism</td>
<td>Zernike</td>
<td>( c_{22} = 0.125 )</td>
</tr>
<tr>
<td>coma</td>
<td>Seidel</td>
<td>( a_{31} = 0.125 )</td>
</tr>
<tr>
<td>coma</td>
<td>Zernike</td>
<td>( c_{31} = 0.125 )</td>
</tr>
</tbody>
</table>

a) optimal constructive interference
b) reduced constructive interference due to phase aberrations
c) reduced effect of phase error by apodization and lower energetic weighting
d) start of destructive interference for 90° or \( \lambda/4 \) phase aberration begin of negative z-component
Criteria of Rayleigh and Marechal

- **Rayleigh criterion:**
  1. maximum of wave aberration: \( W_{pv} < \lambda/4 \)
  2. beginning of destructive interference of partial waves
  3. limit for being diffraction limited (definition)
  4. as a PV-criterion rather conservative: maximum value only in 1 point of the pupil
  5. different limiting values for aberration shapes and definitions (Seidel, Zernike,...)

- **Marechal criterion:**
  1. Rayleigh criterion corresponds to \( W_{rms} < \lambda/14 \) in case of defocus

\[
W_{rms}^{Rayleigh} \leq \frac{\lambda}{\sqrt{192}} = \frac{\lambda}{13.856} \approx \frac{\lambda}{14}
\]

  2. generalization of \( W_{rms} < \lambda/14 \) for all shapes of wave fronts
  3. corresponds to Strehl ratio \( D_s > 0.80 \) (in case of defocus)
  4. more useful as PV-criterion of Rayleigh
PV and $W_{\text{rms}}$-Values

- PV and $W_{\text{rms}}$ values for different definitions and shapes of the aberrated wavefront

- Due to mixing of lower orders in the definition of the Zernikes, the $W_{\text{rms}}$ usually is smaller in comparison to the corresponding Seidel definition

<table>
<thead>
<tr>
<th>aberration type</th>
<th>definition</th>
<th>mean $W_{\text{rms}}$</th>
<th>peak-valley $W_{\text{PV}}$</th>
<th>root mean square $W_{\text{rms}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus</td>
<td>$a_{20} \cdot r_p^2$</td>
<td>$a_{20}$</td>
<td>$\frac{a_{20}}{2}$</td>
<td>$\frac{2a_{20}}{2\sqrt{3}} = 0.289 \cdot a_{20}$</td>
</tr>
<tr>
<td>defocus</td>
<td>$c_{20} \cdot (2r_p^2 - 1)$</td>
<td>0</td>
<td>$2c_{20}$</td>
<td>$\frac{c_{20}}{2\sqrt{3}} = 0.577 \cdot c_{20}$</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>$a_{40} \cdot r_p^4$</td>
<td>$\frac{a_{40}}{3}$</td>
<td>$a_{40}$</td>
<td>$\frac{2a_{40}}{3\sqrt{5}} = 0.298 \cdot a_{40}$</td>
</tr>
<tr>
<td>spherical aberration with defocus</td>
<td>$b_{40} \cdot (r_p^4 - r_p^2)$</td>
<td>$-\frac{b_{40}}{6}$</td>
<td>$\frac{b_{40}}{4}$</td>
<td>$\frac{b_{40}}{6\sqrt{5}} = 0.075 \cdot b_{40}$</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>$c_{40} \cdot (6r_p^4 - 6r_p^2 + 1)$</td>
<td>0</td>
<td>$\frac{3c_{40}}{\sqrt{5}}$</td>
<td>$\frac{c_{40}}{\sqrt{5}} = 0.447 \cdot c_{40}$</td>
</tr>
<tr>
<td>astigmatism</td>
<td>$a_{22} r_p^2 \cos^2 \theta$</td>
<td>$\frac{a_{22}}{4}$</td>
<td>$a_{22}$</td>
<td>$\frac{a_{22}}{4} = 0.25 \cdot a_{22}$</td>
</tr>
<tr>
<td>astigmatism with defocus</td>
<td>$b_{22} \left( r_p^2 \cos^2 \theta - \frac{1}{2} r_p^2 \right)$</td>
<td>0</td>
<td>$b_{22}$</td>
<td>$\frac{b_{22}}{2\sqrt{6}} = 0.204 \cdot b_{22}$</td>
</tr>
<tr>
<td>astigmatism</td>
<td>$c_{22} \left( 2r_p^2 \cos^2 \theta - r_p^2 \right)$</td>
<td>0</td>
<td>$2c_{22}$</td>
<td>$\frac{c_{22}}{\sqrt{6}} = 0.408 \cdot c_{22}$</td>
</tr>
<tr>
<td>coma</td>
<td>$a_{31} r_p^3 \cos \theta$</td>
<td>0</td>
<td>$2a_{31}$</td>
<td>$\frac{a_{31}}{2\sqrt{2}} = 0.353 \cdot a_{31}$</td>
</tr>
<tr>
<td>coma with tilt</td>
<td>$b_{31} \left( r_p^3 - \frac{2}{3} r_p \right) \cos \theta$</td>
<td>0</td>
<td>$\frac{2b_{31}}{3}$</td>
<td>$\frac{b_{31}}{6\sqrt{2}} = 0.118 \cdot b_{31}$</td>
</tr>
<tr>
<td>coma</td>
<td>$c_{31} \left( 3r_p^3 - 2r_p \right) \cos \theta$</td>
<td>0</td>
<td>$2c_{31}$</td>
<td>$\frac{c_{31}}{2\sqrt{2}} = 0.353 \cdot c_{31}$</td>
</tr>
</tbody>
</table>
Typical Variation of Wave Aberrations

- Microscopic objective lens

- Changes of rms value of wave aberration with
  1. wavelength
  2. field position

- Common practice:
  1. diffraction limited on axis for main part of the spectrum
  2. Requirements relaxed in the outer field region
  3. Requirement relaxed at the blue edge of the spectrum
• Rms of the wavefront changes with field position scaled in $\lambda$.

• Only one scalar number per field point: fine structure suppressed.
Zernike Calculation on distorted grids

- Conventional calculation of the Zernikes: equidistant grid in the entrance pupil
- Real systems: Pupil aberrations and distorted grid in the exit pupil
- Deviating positions of phase gives errors in the Zernike calculation
- Additional effect: re-normalization of maximum radius
- 5% radial distorted grid with both signs: 
  +5%: pincushion
  -5% barrel

- Wavefront of only one selected Zernike is re-analyzed on distorted re-normalized grid:

- Effects of distortion:
  - errors in the range of some percent
  - cross-mixing to other zernikes of same symmetry
  - neighboring Zernikes partly influenced until 20%
  - larger effects of higher order Zernikes
  - slightly larger effects for pincushion distortion
Zernike Coefficients for Change of Radius

- Change of normalization radius,
  Problem, if pupil edge is not well known or badly defined

- Deviation in the radius of normalization of the pupil size:
  1. wrong coefficients
  2. mixing of lower orders during fit-calculation, symmetry-dependent

- Example primary spherical aberration:
  polynomial:
  \[ Z_9(r) = 6r^4 - 6r^2 + 1 \]

  Stretching factor \( \varepsilon \) of the radius
  \[ r = \varepsilon \cdot \rho \]

  New Zernike expansion on basis of \( r \)
  \[
  Z_9(r) = \frac{1}{\varepsilon^4} \cdot Z_9(r) + \frac{3(1-\varepsilon^2)}{\varepsilon^4} \cdot Z_4(r)
  + \frac{\varepsilon^4 - 3\varepsilon^2 + 2}{\varepsilon^4} \cdot Z_1
  \]
Zernike Expansion of Local Deviations

Small Gaussian bump in the topology of a surface

N = 36                  N = 64                  N = 100                  N = 144              N = 225                 N = 324                 N = 625
Rms = 0.0237            0.0193                   0.0149                   0.0109                 0.00624               0.00322               0.00047
PV = 0.378              0.307                     0.235                     0.170                   0.0954                  0.0475                  0.0063

Spectrum of coefficients for the last case
Measurement of Wave Aberrations

- Wave aberrations are measurable directly
- Good connection between simulation/optical design and realization/metrology
- Direct phase measuring techniques:
  1. Interferometry
  2. Hartmann-Shack
  3. Hartmann sensor
  4. Special: Moire, Holography, phase-space analyzer
- Indirect measurement by inversion of the wave equation:
  1. Phase retrieval of PSF z-stack
  2. Retrieval of edge or line images
- Indirect measurement by analyzing the imaging conditions: from general image degradation
- Accuracy:
  1. \( \lambda/1000 \) possible, \( \lambda/100 \) standard for rms-value
  2. Rms vs. individual Zernike coefficients
Testing with Twyman-Green Interferometer

- Short common path, sensible setup
- Two different operation modes for reflection or transmission
- Always factor of 2 between detected wave and component under test

1. mode:
lens tested in transmission
auxiliary mirror for auto-collimation

2. mode:
surface tested in reflection
auxiliary lens to generate convergent beam
Interferograms of Primary Aberrations

Spherical aberration $1 \lambda$

Astigmatism $1 \lambda$

Coma $1 \lambda$

Defocussing in $\lambda$
Interferogram - Definition of Boundary

- Critical definition of the interferogram boundary and the Zernike normalization radius in reality
Hartmann Shack Wavefront Sensor

- Lenslet array divides the wavefront into subapertures
- Every lenslet generates a single spot in the focal plane
- The averaged local tilt produces a transverse offset of the spot center
- Integration of the derivative matrix delivers the wave front $W(x,y)$
Hartmann Shack Wavefront Sensor

- Typical setup for component testing

- Lenslet array

![Diagram of Hartmann Shack Wavefront Sensor](image)

- **a)** Subaperture
- **b)** Point spread function
- 2-dimensional lenslet array