Design and Correction of Optical Systems

Lecture 6: Wave Aberrations

2013-05-22

Herbert Gross
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<td>1</td>
<td>10.04</td>
<td>Basics</td>
<td>Law of refraction, Fresnel formulas, optical system model, raytrace, calculation approaches</td>
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<td>2</td>
<td>17.04</td>
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<td>Dispersion, anomalous dispersion, glass map, liquids and plastics, lenses, mirrors, aspheres, diffractive elements</td>
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<td>08.05</td>
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<td>6</td>
<td>22.05</td>
<td>Wave Aberrations</td>
<td>Fermat principle and Eikonal, wave aberrations, expansion and higher orders, Zernike polynomials, measurement of system quality</td>
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<td>PSF and Transfer function</td>
<td>Diffraction, point spread function, PSF with aberrations, optical transfer function, Fourier imaging model</td>
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<td>8</td>
<td>05.06</td>
<td>Further Performance Criteria</td>
<td>Rayleigh and Marechal criteria, Strehl definition, 2-point resolution, MTF-based criteria, further options</td>
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<tr>
<td>9</td>
<td>12.06</td>
<td>Optimization and Correction</td>
<td>Principles of optimization, initial setups, constraints, sensitivity, optimization of optical systems, global approaches</td>
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<tr>
<td>10</td>
<td>19.06</td>
<td>Correction Principles I</td>
<td>Symmetry, lens bending, lens splitting, special options for spherical aberration, astigmatism, coma and distortion, aspheres</td>
</tr>
<tr>
<td>11</td>
<td>26.06</td>
<td>Correction Principles II</td>
<td>Field flattening and Petzval theorem, chromatical correction, achromate, apochromate, sensitivity analysis, diffractive elements</td>
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<tr>
<td>12</td>
<td>03.07</td>
<td>Optical System Classification</td>
<td>Overview, photographic lenses, microscopic objectives, lithographic systems, eyepieces, scan systems, telescopes, endoscopes</td>
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<td>13</td>
<td>10.07</td>
<td>Special System Examples</td>
<td>Zoom systems, confocal systems</td>
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Contents

1. Rays and wavefronts
2. Wave aberrations
3. Expansion of the wave aberrations
4. Zernike polynomials
5. Performance criteria
6. Non-circular pupil areas
7. Measurement of wave aberrations
Law of Malus-Dupin

- Law of Malus-Dupin:
  - equivalence of rays and wavefronts
  - both are orthonormal
  - identical information

- Condition:
  No caustic of rays

- Mathematical:
  Rotation of Eikonal vanish

\[ \text{rot}(n \cdot \vec{s}) = 0 \]

- Optical system:
  Rays and spherical waves orthonormal
Fermat Principle

- Fermat principle: the light takes the ray path, which corresponds to the shortest time of arrival.

- The realized path is a minimum and therefore the first derivatives vanish

\[ \delta L = \delta \int_{P_1}^{P_2} n(x, y, z) \, ds = 0 \]

- Several realized ray pathes have the same optical path length

\[ L = \int_{P_1}^{P_2} n \cdot \vec{s} \cdot d\vec{r} = \text{const.} \]

- The principle is valid for smooth and discrete index distributions.
Ray-Wave Equivalent

- Rays and waves carry the same information
- Wave surface is perpendicular on the rays
- Wave is purely geometrical and has no diffraction properties
Relationships

- Concrete calculation of wave aberration: addition of discrete optical path lengths (OPL)

- Reference on chief ray and reference sphere (optical path difference)

- Relation to transverse aberrations

- Conversion between longitudinal transverse and wave aberrations

- Scaling of the phase / wave aberration:
  1. Phase angle in radiant
  2. Light path (OPL) in mm
  3. Light path scaled in $\lambda$

$$l_{OPL} = \int_{OE}^{AP} n \cdot d\vec{r}$$

$$\Delta_{OPD}(x, y) = l_{OPL}(x, y) - l_{OPL}(0,0)$$

$$\frac{\partial W}{\partial y_p} = - \frac{\Delta y'}{R - W} \approx - \frac{\Delta y'}{R}$$

$$\Delta s' = \frac{R}{y_p} \cdot \Delta y' = \Delta y' = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p}$$

$$E(x) = A(x) \cdot e^{i \varphi(x)}$$

$$E(x) = A(x) \cdot e^{i k \Delta_{OPD}(x)}$$

$$E(x) = A(x) \cdot e^{2 \pi i W(x)}$$
Relationship to Transverse Aberration

- Relation between wave and transverse aberration
- Approximation for small aberrations and small aperture angles $u$
- Ideal wavefront, reference sphere: $W_{\text{ideal}}$
- Real wavefront: $W_{\text{real}}$
- Finite difference

\[ \Delta W = W_{\text{real}} - W_{\text{ideal}} \]

\[ \varphi \approx \tan \varphi = \frac{\partial W}{\partial y_p} \]

\[ \Delta y' = -R \cdot \varphi \]

\[ \frac{\partial W}{\partial y_p} = - \frac{\Delta y'}{R - W} \approx - \frac{\Delta y'}{R} \]
Wave Aberration in Optical Systems

- Definition of optical path length in an optical system:
  Reference sphere around the ideal object point through the center of the pupil
- Chief ray serves as reference
  Difference of OPL : optical path difference OPD
- Practical calculation: discrete sampling of the pupil area, real wave surface represented as matrix
Wave Aberration

- Definition of the peak valley value $W_{PV}$
- Reference sphere corresponds to perfect imaging
- Rms-value is more relevant for performance evaluation
Pupil Sampling

- All rays start in one point in the object plane
- The entrance pupil is sampled equidistant
- In the exit pupil, the transferred grid may be distorted
- In the image plane a spreaded spot diagram is generated
Wave Aberration Criteria

- Mean quadratic wave deviation (\( W_{\text{Rms}} \), root mean square)

\[
W_{\text{rms}} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \sqrt{\frac{1}{A_{\text{ExP}}} \iint [W(x_p, y_p) - W_{\text{mean}}(x_p, y_p)]^2 dx_p dy_p}
\]

with pupil area

\[
A_{\text{ExP}} = \iint dxdy
\]

- Peak valley value \( W_{pv} \): largest difference

\[
W_{pv} = \max \left[ W_{\text{max}}(x_p, y_p) - W_{\text{min}}(x_p, y_p) \right]
\]

- General case with apodization:
  weighting of local phase errors with intensity, relevance for psf formation

\[
W_{\text{rms}} = \sqrt{\frac{1}{A_{\text{ExP}}^{(w)}} \iint I_{\text{ExP}}(x_p, y_p) \cdot [W(x_p, y_p) - W_{\text{mean}}^{(w)}(x_p, y_p)]^2 dx_p dy_p}
\]
Wave Aberrations – Sign and Reference

- Wave aberration: relative to reference sphere
- Choice of offset value: vanishing mean
- Sign of $W$:
  - $W > 0$: stronger convergence intersection: $s < 0$
  - $W < 0$: stronger divergence intersection: $s < 0$

\[
\langle W(x, y) \rangle = \frac{1}{F_{Exp}} \iint W(x, y) \, dx \, dy = 0
\]
Tilt of Wavefront

- Change of reference sphere:
  tilt by angle $\theta$
  linear in $y_p$
  \[ \Delta W_{\text{tilt}} = n \cdot y_p \cdot \theta \]

- Wave aberration due to transverse aberration $\Delta y'$
  \[ \Delta W_{\text{tilt}} = -\frac{y_p}{R_{\text{Ref}}} \cdot \Delta y' \]

- Is the usual description of distortion
Paraxial defocussing by $\Delta z$:
Change of wavefront

$$\Delta W_{\text{Def}} = -\frac{n \cdot r_p^2}{2R_{\text{ref}}^2} \cdot \Delta z' = -\frac{1}{2} n \cdot \Delta z' \cdot \sin^2 u$$
Special Cases of Wave Aberrations

- Wave aberrations are usually given as reduced aberrations:
  - wave front for only 1 field point
  - field dependence represented by discrete cases

- Special case of aberrations:
  1. axial color and field curvature:
     represented as defocussing term, Zernike $c_4$
  2. distortion and lateral color:
     represented as tilt term, Zernike $c_2, c_3$
3. afocal system
   - exit pupil in infinity
   - plane wave as reference

4. telecentric system
   chief ray parallel to axis
Expansion of the Wave Aberration

- Table as function of field and aperture
- Selection rules: checkerboard filling of the matrix

<table>
<thead>
<tr>
<th>Aperture r</th>
<th>Field y</th>
<th>Spherical</th>
<th>Coma</th>
<th>Astigmatism</th>
<th>Distortion</th>
<th>Primary aberrations / Seidel</th>
</tr>
</thead>
<tbody>
<tr>
<td>r&lt;sup&gt;1&lt;/sup&gt;</td>
<td>Distortion</td>
<td>r&lt;sup&gt;1&lt;/sup&gt;</td>
<td>y&lt;sup&gt;0&lt;/sup&gt;</td>
<td>y&lt;sup&gt;1&lt;/sup&gt;</td>
<td>y&lt;sup&gt;2&lt;/sup&gt;</td>
<td>y&lt;sup&gt;3&lt;/sup&gt;</td>
</tr>
<tr>
<td>r&lt;sup&gt;2&lt;/sup&gt;</td>
<td>r&lt;sup&gt;2&lt;/sup&gt;</td>
<td>Defocus</td>
<td></td>
<td></td>
<td>y&lt;sup&gt;2&lt;/sup&gt;r&lt;sup&gt;2&lt;/sup&gt;cos&lt;sup&gt;2&lt;/sup&gt;θ</td>
<td>y&lt;sup&gt;2&lt;/sup&gt;r&lt;sup&gt;2&lt;/sup&gt;cos&lt;sup&gt;2&lt;/sup&gt;θ</td>
</tr>
<tr>
<td>r&lt;sup&gt;3&lt;/sup&gt;</td>
<td>r&lt;sup&gt;3&lt;/sup&gt;</td>
<td></td>
<td></td>
<td></td>
<td>y&lt;sup&gt;3&lt;/sup&gt;r&lt;sup&gt;3&lt;/sup&gt;cos&lt;sup&gt;3&lt;/sup&gt;θ</td>
<td>y&lt;sup&gt;3&lt;/sup&gt;r&lt;sup&gt;3&lt;/sup&gt;cos&lt;sup&gt;3&lt;/sup&gt;θ</td>
</tr>
<tr>
<td>r&lt;sup&gt;4&lt;/sup&gt;</td>
<td>r&lt;sup&gt;4&lt;/sup&gt;</td>
<td>Spherical primary</td>
<td>y&lt;sup&gt;4&lt;/sup&gt;r&lt;sup&gt;4&lt;/sup&gt;cos&lt;sup&gt;4&lt;/sup&gt;θ</td>
<td>y&lt;sup&gt;2&lt;/sup&gt;r&lt;sup&gt;4&lt;/sup&gt;cos&lt;sup&gt;2&lt;/sup&gt;θ</td>
<td>y&lt;sup&gt;2&lt;/sup&gt;r&lt;sup&gt;4&lt;/sup&gt;</td>
<td></td>
</tr>
<tr>
<td>r&lt;sup&gt;5&lt;/sup&gt;</td>
<td></td>
<td>y&lt;sup&gt;5&lt;/sup&gt;r&lt;sup&gt;5&lt;/sup&gt;cos&lt;sup&gt;5&lt;/sup&gt;θ</td>
<td>y&lt;sup&gt;2&lt;/sup&gt;r&lt;sup&gt;5&lt;/sup&gt;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r&lt;sup&gt;6&lt;/sup&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>r&lt;sup&gt;6&lt;/sup&gt; Spherical secondary</td>
<td></td>
</tr>
</tbody>
</table>
Polynomial Expansion of Wave Aberrations

- Taylor expansion of the wavefront:
  \[ W(y', r_p, \theta) = \sum_{k,l,m} W_{klm} y'^k r_p^l \cos^m \theta \]

  - \( y' \) Image height \( \text{index } k \)
  - \( r_p \) Pupil height \( \text{index } l \)
  - \( \theta \) Pupil azimuth angle \( \text{index } m \)

- Symmetry invariance:
  1. Image height
  2. Pupil height
  3. Scalar product between image and pupil vector

\[ \lvert \vec{y}' \rvert \]
\[ \lvert \vec{r}_p \rvert \]
\[ \vec{y}' \cdot \vec{r}_p = y' \cdot r_p \cdot \cos \theta \]

- Number of terms
  sum of indices in the exponent \( i_{sum} \)

<table>
<thead>
<tr>
<th>( i_{sum} )</th>
<th>( N_i ) number of terms</th>
<th>Type of aberration</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>2</td>
<td>image location</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>primary aberrations, 4th order</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>secondary aberrations, 6th order</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>8th order</td>
</tr>
</tbody>
</table>
The exponents of the Taylor expansion on the aperture depends on the kind of representation of the aberrations.
- The exponent grows by 1 in the sequence longitudinal-transversal-wave aberrations
- The Seidel term '3rd order' is valid only for transverse aberrations
- Dependence on aperture and field size for the primary aberrations:

<table>
<thead>
<tr>
<th>type of aberration</th>
<th>wave aberration</th>
<th>transverse aberration</th>
<th>longitudinal aberration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>u</td>
<td>w</td>
<td>u</td>
</tr>
<tr>
<td>spherical</td>
<td>4</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>coma</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>astigmatism</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Petzval curvature</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>distortion</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>axial chromatical</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>chromatical magnif.</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Taylor Expansion of the Primary Aberrations

- Expansion of the monochromatic aberrations
- First real aberration: primary aberrations, 4th order as wave deviation

\[ W(y', r_p, y_p) = A_S r_p^4 + A_C y' r_p^2 y_p + A_a y'^2 y_p^2 + A_p y'^2 r_p^2 + A_d y'^3 y_p \]

- Coefficients of the primary aberrations:
  \( A_S \): Spherical Aberration
  \( A_C \): Coma
  \( A_A \): Astigmatism
  \( A_P \): Petzval curvature
  \( A_D \): Distortion

- Alternatively: expansion in polar coordinates:
  Zernike basis expansion, usually only for one field point, orthogonalized
Zernike Polynomials

- Expansion of the wave aberration on a circular area

\[ W(r, \varphi) = \sum_{n} \sum_{m=-n}^{n} c_{nm} Z_n^m(r, \varphi) \]

- Zernike polynomials in cylindrical coordinates:
  Radial function \( R_n^m(r) \), index \( n \)
  Azimuthal function \( \varphi \), index \( m \)

- Orthonormality

- Advantages:
  1. Minimal properties due to \( W_{\text{rms}} \)
  2. Decoupling, fast computation
  3. Direct relation to primary aberrations for low orders

- Problems:
  1. Computation on discrete grids
  2. Non circular pupils
  3. Different conventions concerning indices, scaling, coordinate system
Balance of Lower Orders by Zernike Polynomials

- Mixing of lower orders to get the minimal $W_{rms}$

- Example spherical aberration:
  1. Spherical 4th order according to Seidel
  2. Additional quadratic expression:
     Optimal defocussing for edge correction
  3. Additional absolute term
     Minimale value of $W_{rms}$

$$W(r_p) = 6r_p^4 - 6r_p^2 + 1$$
Zernike Polynomials

- Zernike polynomials orders by indices:
  - \( n \): radial
  - \( m \): azimuthal, \( \sin/cos \)
- Orthonormal function on unit circle

\[
Z_n^m(r, \varphi) = R_n^m(r) \begin{cases} 
\sin m\varphi & \text{für } m > 0 \\
\cos m\varphi & \text{für } m < 0 \\
1 & \text{für } m = 0
\end{cases}
\]

- Expansion of wave aberration surface

\[
W(r, \varphi) = \sum_{n} \sum_{m=0}^{n} c_{nm} Z_n^m(r, \varphi)
\]

- Direct relation to primary aberration types
- Direct measurement by interferometry
- Orthogonality perturbed:
  1. apodization
  2. discretization
  3. real non-circular boundary
Azimuthal Dependence of Zernike Polynomials

- Azimuthal spatial frequency
<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>Polar coordinates</th>
<th>Cartesian coordinates</th>
<th>Interpretation</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>piston</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(r \sin \varphi)</td>
<td>(x)</td>
<td>tilt in x</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>(r \cos \varphi)</td>
<td>(y)</td>
<td>tilt in y</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(r^2 \sin 2 \varphi)</td>
<td>(2xy)</td>
<td>Astigmatism 45°</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(2r^2 - 1)</td>
<td>(2x^2 + 2y^2 - 1)</td>
<td>defocussing</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>(r^2 \cos 2 \varphi)</td>
<td>(y^2 - x^2)</td>
<td>Astigmatism 0°</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(r^3 \cos 2 \varphi)</td>
<td>(y^2 - x^2)</td>
<td>trefoil 30°</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>((3r^3 - 2r) \sin \varphi)</td>
<td>(3x^3 - 2x + 3xy^2)</td>
<td>coma x</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>((3r^3 - 2r) \cos \varphi)</td>
<td>(3y^3 - 2y + 3x^2y)</td>
<td>coma y</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>(r^3 \cos 3 \varphi)</td>
<td>(y^3 - 3x^2y)</td>
<td>trefoil 0°</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>(r^4 \sin 4 \varphi)</td>
<td>(4xy^3 - 4x^3y)</td>
<td>Four sheet 22.5°</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>((4r^4 - 3r^2) \sin 2 \varphi)</td>
<td>(8xy^3 + 8x^3y - 6xy)</td>
<td>Secondary astigmatism</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>(6r^4 - 6r^2 + 1)</td>
<td>(6x^4 + 6y^4 + 12x^2y^2 - 6x^2 - 6y^2 + 1)</td>
<td>Spherical aberration</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
<td>((4r^4 - 3r^2) \cos 2 \varphi)</td>
<td>(4y^4 - 4x^4 + 3x^2 - 3y^2 - 4x^2y^2)</td>
<td>Secondary astigmatism</td>
</tr>
<tr>
<td>4</td>
<td>-4</td>
<td>(r^4 \cos 4 \varphi)</td>
<td>(y^4 + x^4 - 6x^2y^2)</td>
<td>Four sheet 0°</td>
</tr>
</tbody>
</table>
Zernike Polynomials

- Advantages of the Zernike polynomials:
  - de-coupling due to orthogonality
  - stable numerical computation
  - direct relation of lower orders to classical aberrations
  - optimal balancing of lower orders (e.g. best defocus for spherical aberration)
  - fast calculation of $W_{\text{rms}}$ and Strehl ratio in approximation of Marechal

- Necessary requirements for orthogonality:
  - pupil shape circular
  - uniform illumination of pupil (corresponds to constant weighting)
  - no discretization effects (finite number of points, boundary)

- Different standardizations used concerning indexing, scaling, sign of coordinates (orientation for off-axis field points):
  - Fringe representation: peak value 1, normalized
  - Standard representation $W_{\text{rms}}$ normalized
  - Original representation according to Nijbor-Zernike

Norm ISO 10110 allows Fringe and Standard representation
Calculation of Zernike Polynomials

- Assumptions:
  1. Pupil circular
  2. Illumination homogeneous
  3. Neglectible discretization effects /sampling, boundary)

- Direct computation by double integral:
  1. Time consuming
  2. Errors due to discrete boundary shape
  3. Wrong for non circular areas
  4. Independent coefficients

- LSQ-fit computation:
  1. Fast, all coefficients $c_j$ simultaneously
  2. Better total approximation
  3. Non stable for different numbers of coefficients, if number too low
  4. Stable for non circular shape of pupil

\[
c_j = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} W(r, \varphi) Z_j(r, \varphi) \, d\varphi \, dr
\]

\[
\sum_{i=1}^{N} \left[ W_i - \sum_{j=1}^{N} c_j Z_j(r_i) \right]^2 = \min
\]

\[
\bar{c} = \left( Z^T Z \right)^{-1} Z^T \bar{W}
\]
Performance Description by Zernike Expansion

- Vector of \( c_j \) linear sequence with running index

- Sorting by symmetry
Zernike polynomials: Different Conventions

- Different standardizations used concerning:
  1. indexing
  2. scaling / normalization
  3. sign of coordinates (orientation for off-axis field points)

- Fringe - representation
  1. CodeV, Zemax, interferometric test of surfaces
  2. Standardization of the boundary to $\pm 1$
  3. no additional prefactors in the polynomial
  4. Indexing according to m (Azimuth), quadratic number terms have circular symmetry
  5. coordinate system invariant in azimuth

- Standard - representation
  - CodeV, Zemax, Born / Wolf
  - Standardization of rms-value on $\pm 1$ (with prefactors), easy to calculate Strehl ratio
  - coordinate system invariant in azimuth

- Original - Nijboer - representation
  - Expansion:
    \[
    W(r, \varphi) = a_{00} + \frac{1}{\sqrt{2}} \sum_{n=0}^{k} a_{0n} R_n^0 + \sum_{n=0}^{k} \sum_{m=1}^{n} a_{nm} R_n^m \cos(m\varphi) + \sum_{n=0}^{k} \sum_{m=1}^{n} b_{nm} R_n^m \sin(m\varphi)
    \]
  - Standardization of rms-value on $\pm 1$
  - coordinate system rotates in azimuth according to field point
Criteria of Rayleigh and Marechal

- **Rayleigh criterion:**
  1. maximum of wave aberration: \( W_{pv} < \lambda/4 \)
  2. beginning of destructive interference of partial waves
  3. limit for being diffraction limited (definition)
  4. as a PV-criterion rather conservative: maximum value only in 1 point of the pupil
  5. different limiting values for aberration shapes and definitions (Seidel, Zernike,...)

- **Marechal criterion:**
  1. Rayleigh criterion corresponds to \( W_{rms} < \lambda/14 \) in case of defocus

\[
W_{rms}^{Rayleigh} \leq \frac{\lambda}{\sqrt{192}} = \frac{\lambda}{13.856} \approx \frac{\lambda}{14}
\]

  2. generalization of \( W_{rms} < \lambda/14 \) for all shapes of wave fronts
  3. corresponds to Strehl ratio \( D_s > 0.80 \) (in case of defocus)
  4. more useful as PV-criterion of Rayleigh
The Rayleigh criterion \[ |W_{PV}| \leq \frac{\lambda}{4} \]
gives individual maximum aberrations coefficients, depends on the form of the wave.

Examples:

<table>
<thead>
<tr>
<th>aberration type</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus</td>
<td>Seidel</td>
</tr>
<tr>
<td></td>
<td>( a_{20} = 0.25 )</td>
</tr>
<tr>
<td>defocus</td>
<td>Zernike</td>
</tr>
<tr>
<td></td>
<td>( c_{20} = 0.125 )</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Seidel</td>
</tr>
<tr>
<td></td>
<td>( a_{40} = 0.25 )</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Zernike</td>
</tr>
<tr>
<td></td>
<td>( c_{40} = 0.167 )</td>
</tr>
<tr>
<td>astigmatism</td>
<td>Seidel</td>
</tr>
<tr>
<td></td>
<td>( a_{22} = 0.25 )</td>
</tr>
<tr>
<td>astigmatism</td>
<td>Zernike</td>
</tr>
<tr>
<td></td>
<td>( c_{22} = 0.125 )</td>
</tr>
<tr>
<td>coma</td>
<td>Seidel</td>
</tr>
<tr>
<td></td>
<td>( a_{31} = 0.125 )</td>
</tr>
<tr>
<td>coma</td>
<td>Zernike</td>
</tr>
<tr>
<td></td>
<td>( c_{31} = 0.125 )</td>
</tr>
</tbody>
</table>
PV and $W_{rms}$-Values

- PV and $W_{rms}$ values for different definitions and shapes of the aberrated wavefront

- Due to mixing of lower orders in the definition of the Zernikes, the $W_{rms}$ usually is smaller in comparison to the corresponding Seidel definition

<table>
<thead>
<tr>
<th>aberration type</th>
<th>definition</th>
<th>mean $W_{mean}$</th>
<th>peak-valley $W_{pv}$</th>
<th>root mean square $W_{rms}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>defocus</td>
<td>$a_{20} \cdot r_p^2$</td>
<td>$\frac{a_{20}}{2}$</td>
<td>$a_{20}$</td>
<td>$\frac{a_{20}}{2} \cdot \frac{2}{3} = 0.289 \cdot a_{20}$</td>
</tr>
<tr>
<td>defocus</td>
<td>$c_{20} \cdot (2r_p^2 - 1)$</td>
<td>0</td>
<td>$2c_{20}$</td>
<td>$\frac{c_{20}}{\sqrt{3}} = 0.577 \cdot c_{20}$</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Seidel $a_{40} \cdot r_p^4$</td>
<td>$\frac{a_{40}}{3}$</td>
<td>$a_{40}$</td>
<td>$\frac{2a_{40}}{3 \cdot \sqrt{5}} = 0.298 \cdot a_{40}$</td>
</tr>
<tr>
<td>spherical aberration with defocus</td>
<td>$b_{40} \cdot (r_p^4 - r_p^2)$</td>
<td>$-\frac{b_{40}}{6}$</td>
<td>$\frac{b_{40}}{4}$</td>
<td>$\frac{b_{40}}{6 \cdot \sqrt{5}} = 0.075 \cdot b_{40}$</td>
</tr>
<tr>
<td>spherical aberration</td>
<td>Zernike $c_{40} \cdot (6r_p^4 - 6r_p^2 + 1)$</td>
<td>0</td>
<td>$\frac{3c_{40}}{\sqrt{5}} = 0.447 \cdot c_{40}$</td>
<td></td>
</tr>
<tr>
<td>astigmatism</td>
<td>Seidel $a_{22} \cdot r_p^2 \cos^2 \theta$</td>
<td>$\frac{a_{22}}{4}$</td>
<td>$a_{22}$</td>
<td>$\frac{a_{22}}{4} = 0.25 \cdot a_{22}$</td>
</tr>
<tr>
<td>astigmatism with defocus</td>
<td>$b_{22} \left( r_p^2 \cos^2 \theta - \frac{1}{2} r_p^2 \right)$</td>
<td>0</td>
<td>$b_{22}$</td>
<td>$\frac{b_{22}}{2 \cdot \sqrt{6}} = 0.204 \cdot b_{22}$</td>
</tr>
<tr>
<td>astigmatism</td>
<td>Zernike $c_{22} \left( 2r_p^2 \cos^2 \theta - r_p^2 \right)$</td>
<td>0</td>
<td>$2c_{22}$</td>
<td>$\frac{c_{22}}{\sqrt{6}} = 0.408 \cdot c_{22}$</td>
</tr>
<tr>
<td>coma</td>
<td>Seidel $a_{31} \cdot r_p^3 \cos \theta$</td>
<td>0</td>
<td>$2a_{31}$</td>
<td>$\frac{a_{31}}{2 \cdot \sqrt{2}} = 0.353 \cdot a_{31}$</td>
</tr>
<tr>
<td>coma with tilt</td>
<td>$b_{31} \left( r_p^3 - \frac{2}{3} r_p \right) \cos \theta$</td>
<td>0</td>
<td>$2b_{31}$</td>
<td>$\frac{b_{31}}{6 \cdot \sqrt{2}} = 0.118 \cdot b_{31}$</td>
</tr>
<tr>
<td>coma</td>
<td>Zernike $c_{31} \left( 3r_p^3 - 2r_p \right) \cos \theta$</td>
<td>0</td>
<td>$2c_{31}$</td>
<td>$\frac{c_{31}}{2 \cdot \sqrt{2}} = 0.353 \cdot c_{31}$</td>
</tr>
</tbody>
</table>
Typical Variation of Wave Aberrations

- Microscopic objective lens

- Changes of rms value of wave aberration with
  1. wavelength
  2. field position

- Common practice:
  1. diffraction limited on axis for main part of the spectrum
  2. Requirements relaxed in the outer field region
  3. Requirement relaxed at the blue edge of the spectrum

![Graph showing changes of rms value with respect to wavelength and field position.](image)
- Orthogonalization of Zernike Polynomials for ring shaped pupil area

- Basis function depends on obsuration parameter $e$: no easy comparisons possible
• 2D-Legendre polynomials for rectangular areas

• Application:
  Spectrometer slit aperture

• First few polynomials:
  quite similar to Zernikes
Measurement of Wave Aberrations

- Wave aberrations are measurable directly
- Good connection between simulation/optical design and realization/metrology
- Direct phase measuring techniques:
  1. Interferometry
  2. Hartmann-Shack
  3. Hartmann sensor
  4. Special: Moire, Holography, phase-space analyzer
- Indirect measurement by inversion of the wave equation:
  1. Phase retrieval of PSF z-stack
  2. Retrieval of edge or line images
- Indirect measurement by analyzing the imaging conditions:
  from general image degradation
- Accuracy:
  1. $\lambda/1000$ possible, $\lambda/100$ standard for rms-value
  2. Rms vs. individual Zernike coefficients
Testing with Twyman-Green Interferometer

- Short common path, sensible setup
- Two different operation modes for reflection or transmission
- Always factor of 2 between detected wave and component under test

1. mode: lens tested in transmission auxiliary mirror for auto-collimation

2. mode: surface tested in reflection auxiliary lens to generate convergent beam
Interferograms of Primary Aberrations

Spherical aberration 1 $\lambda$

Astigmatism 1 $\lambda$

Coma 1 $\lambda$

Defocussing in $\lambda$

-1  -0.5  0  +0.5  +1
Problems in real world measurement:
- Edge effects
  - Definition of boundary
- Perturbation by coherent stray light
- Local surface error are not well described by Zernike expansion
- Convolution with motion blur

Ref: B. Dörband
Critical definition of the interferogram boundary and the Zernike normalization radius in reality
Summary of Important Topics

- The wave fronts can describe a light as well as rays
- The basis of wave front description is the optical path length
- The wave aberrations are the deviation against a reference sphere
- A Taylor expansion describes the dependence on field and aperture size
- A Zernike expansion is orthogonal on a unit circle
- The Zernike representation is compact and has some advantages: orthogonality, good matching of geometry for optical systems, fast and accurate calculation
- The lower Zernike terms corresponds to the classical aberrations
- Non-circular pupils require other expansions