4.1 Spherical Aberration of a Plane Parallel Plate

Derive the axial spherical aberration, which corresponds to the difference between the real and the paraxial image location of a plane parallel plate with index $n$ and thickness $d$. Calculate the lowest order of the aberration as a function of a small value of $\sin u$. If the diameter in the gaussian image plane should not be larger than 10 μm, calculate the greatest possible thickness of a plate in this approximation with refractive index of $n = 1.48$ for a numerical aperture of $\sin u = 0.8$.

Solution:

The incidence angle is $i = u$

If the ray height at the front side of the plate is given by $y_1$, and the ray angle $i'$ inside the plate is $n \sin i' = \sin i = \sin u$

$$\sin i' = \frac{\sin u}{n}$$

the height at the rear surface is

$$y_2 = y_1 - d \cdot \tan i' = y_1 - d \cdot \frac{\sin u / n}{\sqrt{1 - (\sin u / n)^2}}$$

and the intersection length of the image plane behind the plate is given by

$$s' = \frac{y_2}{\tan u} = \frac{y_1 - d \cdot \frac{\sin u / n}{\sqrt{1 - (\sin u / n)^2}}}{\frac{\sin u}{\sqrt{1 - \sin^2 u}}}$$

The paraxial approximation is given by
\[ s'_0 = \frac{y_1}{\sin u} - \frac{d}{n} \]

Therefore the difference is the spherical aberration due to the plate

\[ \Delta s' = -\frac{d}{n} \cdot \left( \frac{1}{\sqrt{1 - \sin^2 u}} - 1 \right) \]

By Taylor expansion we get the expression

\[ \Delta s' = -\frac{d}{n} \cdot \left( 1 - \frac{1}{2} \sin^2 u \right) \cdot \left( 1 + \frac{1}{2n^2} \sin^2 u \right) \]

\[ = -\frac{d}{n} \cdot \sin^2 u \cdot \left( \frac{1}{2} + \frac{1}{2n^2} \right) = -\frac{d}{n} \cdot \frac{(n^2 - 1)}{2n^3} \cdot \sin^2 u \]

The diameter of the lateral blurr spot is given by

\[ D = 2 \cdot \Delta s' \cdot \sin u = \frac{d \cdot (n^2 - 1)}{n^3} \cdot \sin^3 u \]

Therefore we have for the allowed thickness of the plate

\[ d = \frac{D}{(n^2 - 1) \cdot \left( \frac{n}{\sin u} \right)^3} = 0.053 \text{ mm} \]

### 4.2 Spherical Aberration of a single lens

A thin plane-convex lens with focal length \( f' = 100 \text{ mm} \) is used to focus a collimated beam with diameter \( D = 10 \text{ mm} \) and wavelength \( \lambda = 1.06 \mu \text{m} \).

Draw a sketch of the lens in the optimal setup and explain, why this orientation is advantageous. Calculate the magnification parameter \( M \) and the bending parameter \( X \) of the lens.

The surface contribution of the primary spherical wave aberration of a single lens can be expressed by the formula

\[ A_s = \frac{1}{32n(n-1)f^3} \left[ \frac{n^3}{n-1} + \frac{n+2}{n-1} \left\{ X - \frac{2(n^2-1)}{n+2} M \right\}^2 - \frac{n^2(n-1)}{n+2} M^2 \right] \]

Calculate this coefficient for the refractive indices \( n = 1.4 \) and \( n = 2.0 \). Which choice of refractive index is more advantageous?

Calculate the transverse aberration \( \Delta y' \) in the image plane in the one-dimensional case with the pupil coordinate \( y_p \) by using this formula in the form \( W_{sp} \left( y_p \right) = A_s \cdot y_p^4 \) for both indices. Is the system diffraction limited? Compare the geometrical spot size and the diffraction Airy diameter.

**Solution:**

Sketch of the setup:
In this orientation, the bending of the rays is distributed on two surfaces. This gives smaller incidence angles and a smaller impact of the non-linearity of the law of refraction, which generates aberrations.

The bending parameter is: \( X = +1 \)
the magnification parameter is: \( M = +1 \)

By inserting the data for \( f \), \( n \), \( X \) and \( M \) we get the coefficients:

1. index \( n = 1.4 \) : \( A_s = -4.598 \times 10^{-7} \text{ mm}^{-3} \)
2. index \( n = 2.0 \) : \( A_s = -1.25 \times 10^{-7} \text{ mm}^{-3} \)

The transverse aberration is related by with the wave aberration by

\[
\Delta y' = -R \cdot \frac{\partial W}{\partial y_p} = -f' \cdot 4y_p^3 \cdot A_s
\]

with reference sphere radius \( R = f' \). This gives for the two cases the geometrical spot sizes

\( n = 1.4 \quad D_{\text{Geo}} = 2\Delta y' = 46 \mu m \)
\( n = 2.0 \quad D_{\text{Geo}} = 2\Delta y' = 12.5 \mu m \)

The Airy diameter is given by

\[
D_{\text{Airy}} = \frac{1.22 \cdot \lambda}{\sin u'} \cdot \frac{2.44 \cdot \lambda \cdot f'}{\varnothing_{\text{in}}} = 25.9 \mu m
\]

Therefore, in the case of the higher index \( n=2 \), the system is diffraction limited, the smaller index delivers a geometrical dominated broadening of the spot size.

### 4.3 Wave Aberrations I

The wave aberration of spherical aberration of 5th order reads in polar coordinates on a normalized pupil radius \( W(r) = a_6 \cdot r^6 \) with coefficient \( a_6 \). Calculate the rms-value of the wave aberration. If the deviations are partly compensated by a defocus term, the wave aberration reads \( W(r) = c_6 \cdot (r^6 - r^3) \).

Calculate the mean value and the rms value for this aberration function. Determine the radius \( r_{\text{min}} \) of the pupil radius, where the aberration of \( W \) has its largest value. What is the value of the wave aberration at half of the pupil radius?

**Solution**

Mean value

\[
\langle W \rangle = \frac{1}{\pi} a_6 \int_0^1 r^6 2\pi r dr = \frac{a_6}{\pi} \left[ \frac{2\pi r^8}{8} \right]_0^1 = \frac{a_6}{4}
\]

mean square

\[
\langle W^2 \rangle = \frac{1}{\pi} a_6^2 \int_0^1 r^{12} 2\pi r dr = \frac{a_6^2}{7}
\]
rms value
$$W_{rms} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \frac{3 \cdot a_6}{4 \sqrt{7}} = 0.2835 \cdot a_6$$

Formulation with defocus term
$$W(r) = c_6 \cdot \left( r^6 - r^2 \right)$$

rms-value, calculation as above for the quadratic term
$$\langle W \rangle = \frac{1}{\pi} c_6 \int_0^1 (r^6 - r^2) 2\pi r dr = \frac{c_6}{\pi} \left[ \frac{2\pi r^8}{8} \right]_0^1 - \frac{c_6}{\pi} \left[ \frac{2\pi r^4}{4} \right]_0^1 = c_6 \cdot \left( \frac{1}{4} - \frac{1}{2} \right) = -\frac{c_6}{4}$$

$$\langle W^2 \rangle = \frac{1}{\pi} c_6^2 \int_0^1 (r^6 - r^2)^2 2\pi r dr = 2c_6^2 \int_0^1 r^{13} - 2r^9 + r^5 dr = 2c_6^2 \left( \frac{1}{14} - \frac{1}{5} + \frac{1}{6} \right) = \frac{8}{105} \cdot c_6^2$$

rms value
$$W_{rms} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \sqrt{\frac{8}{105} - \frac{1}{16}} \cdot c_6 = \sqrt{\frac{23}{1680}} \cdot c_6 = 0.117 \cdot c_6$$

Derivative for the calculation of the minimum
$$\frac{\partial W}{\partial r} = c_6 \cdot \left( 6r^5 - 2r \right) = 2r \cdot c_6 \cdot \left( 3r^4 - 1 \right) = 0$$

radius location of the minimum
$$r = \pm \frac{1}{\sqrt{3}} = \pm 0.760$$

In the middle of the pupil radius we have the aberration
$$W\left( \frac{1}{2} \right) = c_6 \cdot \left( r^6 - r^2 \right) = c_6 \cdot \left( \frac{1}{64} - \frac{1}{4} \right) = -c_6 \cdot \frac{15}{64} = -0.234 \cdot c_6$$

### 4.4 Zernike Polynomials

The usage of Zernike polynomials for the quality assessment of optical systems is based on several assumptions and constraints. If these are violated, the usefulness of the Zernike coefficients is decreased. If the benefit is described in three categories as
A: very useful, good criterion
B: limited advantage, but still possible
C: not useful, it makes no sense to use Zernikes as criterion

Rate the following 6 arrangements of optical systems concerning the usage of Zernike polynomials:

1. Circular pupil, gaussian apodization with width $2w_{gauss} = D_{aperture}$ ($2w_{gauss}$ is the $1/e^2$ diameter of the gaussian profile)
2. Circular pupil, uniformly illuminated pupil
3. Pupil with circular ring shape, 10% of area central obscuration, uniform illumination
4. Slit aperture with diameters $D_x/D_y = 1:10$, uniform illumination
5. Square aperture, parabolic apodization with edge value 90% of the central value
6. Square aperture, gaussian apodization with $6w_{gauss} = D_{aperture}$

Solution:

1. Circular pupil, gaussian apodization with width $2w_{gauss} = D_{aperture}$ \hspace{2cm} B
2. Circular pupil, uniform illuminated \hspace{2cm} A
3. Pupil with circular ring shape, 10% central obscuration, uniform illumination  
4. Slit aperture with diameters $D_x/D_y = 1:10$, uniform illumination  
5. Square aperture, parabolic apodization with edge value 80% of the central value  
6. Square aperture, gaussian apodization with $6w_{\text{gauss}} = D_{\text{aperture}}$

### 4.5 Wave aberrations II

Consider a wavefront aberration with the equation

$$W(r) = 0.5 \cdot r^6 + c_2 \cdot r^2$$

with a normalized radial pupil coordinate $r$.

a) What is the type of aberration described by this equation?

b) Draw a sketch of this wavefront for the special value $c_2 = -0.5$.

c) Calculate the rms-value of the aberration for general values of $c_2$.

d) What is the value of $c_2$, for which the rms-value reaches its minimum?

e) What is the range of values for $c_2$, for which the Strehl ratio in Marechal approximation is better than 50%?

**Solution:**

a) Type: spherical aberration 5th order (zone aberration) with defocus

b) 

![Wavefront Sketch](image)

c) Rms value:

\[
\langle W \rangle = \frac{1}{\pi} \int_0^1 \left( \frac{1}{2} r^6 + c_2 r^2 \right) 2\pi r dr = \frac{1}{8} + \frac{1}{2} c_2
\]

\[
\langle W^2 \rangle = \frac{1}{\pi} \int_0^1 \left( \frac{1}{4} r^{12} + c_2 r^8 + c_2^2 r^4 \right) 2\pi r dr = \frac{1}{28} + \frac{1}{5} c_2 + \frac{1}{3} c_2^2
\]

\[
W_{\text{rms}} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2} = \sqrt{\frac{9}{448} + \frac{3}{40} c_2 + \frac{1}{12} c_2^2}
\]

d) Condition for a minimum:

\[
\frac{dW_{\text{rms}}^2}{dc_2} = \frac{3}{40} + \frac{1}{6} c_2 = 0
\]

Therefore

\[
c_2 = -\frac{9}{20}
\]

e) The Marechal approximation reads

\[
D = 1 - 4\pi^2 W_{\text{rms}}^2 = 0.5
\]

With the equation above we get
\[
3.290 \cdot \frac{c_2^2}{\lambda^2} + 2.961 \cdot \frac{c_2}{\lambda^2} - \frac{0.2931}{\lambda^2} = 0
\]
there are two exact solutions
\[
\begin{pmatrix}
c_2 \\
\lambda
\end{pmatrix} = \begin{cases} 
0.990 \\
0.090
\end{cases}
\]
In between these limits, \( D_s < 0.5 \) is fulfilled.