Exercise 2-1: Focal Length of a Thick Lens

Use the formulas of the paraxial raytrace procedure
\[ y_j = y_{j-1} - d_{j-1} U_{j-1} \]
\[ i_j = c_j y_j - U_{j-1} \quad i'_j = \frac{n_j}{n_j'} i_j \]
\[ U'_j = U_j + i_j - i'_j \]
to derive the equation for the focal length of a thick lens in air.

Solution:

An incoming ray parallel to the optical axis with height \( y \) is considered. The paraxial set of formulas is applied as follows: First surface:
\[ U_i = 0 \quad i_i = c_i y \]
\[ i'_i = \frac{1}{n'} i_i \quad U'_i = c_i y - \frac{c_i y}{n} \]
Second surface:
\[ y_2 = y - d \left( c_1 y - \frac{c_2 y}{n} \right) \]

\[ i_2 = c_2 y_2 - U' = c_2 y_2 - d \cdot c_1 c_2 y_2 \frac{n-1}{n} - c_1 y_1 \frac{n-1}{n} \]

\[ i''_2 = n i_2 = n c_2 y_2 - n c_1 y_1 (n-1) + d \cdot y \cdot c_1 c_2 \frac{(n-1)^2}{n} \]

\[ U''_2 = y_2 c_2 - i''_2 = y \left[ c_1 (n-1) - c_2 (n-1) + d \cdot c_1 c_2 \frac{(n-1)^2}{n} \right] \]

From the definition of the focal length

\[ f = \frac{y}{U''_2} \]

we get

\[ \frac{1}{f} = \frac{n-1}{R_1} - \frac{n-1}{R_2} + d \cdot \frac{(n-1)^2}{R_1 R_2 n} \]
Exercise 2-2: Axial Image Shift in a Medium

Calculate the axial shift of the image position, if a ray bundle with aperture \( \sin(u) \) in a medium with refractive index \( n \) is focused into a medium with index \( n' \) exact and in paraxial approximation.

Solution:

The incidence angle is \( i = u \)

If the ray height at the interface plane is given by \( y \), the intersection length without the medium is

\[
s = \frac{y}{\tan u}
\]

In reality the ray is refracted and with the law of refraction \( n \sin i = n' \sin i' \)

we get correspondingly

\[
s' = \frac{y}{\tan u'}
\]

Therefore we obtain the difference as

\[
\Delta s = s' - s = \frac{y}{\tan u'} - \frac{y}{\tan u} = \frac{y}{\tan u} \left[ \frac{\sqrt{1 - \sin^2 i'}}{\sin i} - \frac{\sqrt{1 - \sin^2 u}}{\sin u} \right]
\]

\[
= y \cdot \left[ \frac{\sqrt{1 - (n/n' \sin u)^2}}{n/n' \sin u} - \frac{\sqrt{1 - \sin^2 u}}{\sin u} \right]
\]

In the paraxial approximation we get

\[
\Delta s = -\frac{y \cdot (n - n')}{n \sin u}
\]
Exercise 2-3: Lagrange Invariant for Illumination System

An object with 2.5 mm diameter should be illuminated with a numerical aperture of \( NA = 0.3 \). If the aplanatic corrected illumination system can accept a numerical aperture of \( NA = 0.9 \) of the light source, what is the minimum size of the radiating area of the lamp?

Solution:

According to the constant Lagrange invariant or the fulfillment of the sine condition we have the relation between source and object space

\[
y \cdot n \cdot \sin u = y' \cdot n' \cdot \sin u'
\]

\[
D_{ill} \cdot NA_{ill} = D_{obj} \cdot NA_{obj}
\]

Therefore the light source should at least have a diameter of

\[
D_{ill} = \frac{NA_{obj}}{NA_{ill}} \cdot D_{obj} = \frac{0.3}{0.9} \cdot 2.5\, mm = 0.8333\, mm
\]
Exercise 2-4: Defocussed Telescope

An inverted afocal Galilean telescope is given which reduces the diameter of an incoming beam by a factor of 5. Both used lenses have one plane surface, the positive lens with \( f_1 = 100 \text{ mm} \) is made of a material with refractive index \( n_1 = 1.5 \), the negative lens with \( f_2 = -20 \text{ mm} \) has \( n_2 = 2.0 \).

Sketch the system with an orientation of the lenses, which is beneficial for the correction. Determine the bending and the magnification parameter of both lenses. Calculate the paraxial ABCD-matrix of the system for an arbitrary distance \( z \) between the lenses.

If the distance \( z \) between both lenses is misadjusted by \( \Delta z \), the system is no longer afocal. Calculate the change of the magnification and the residual outgoing ray angle for the case of a misadjustment of \( \Delta = 0.1 \text{ mm} \) and an incoming beam diameter of 20 mm.

Solution:

The lenses have to be oriented in the way shown above. Both ray bendings are then distributed over both surfaces.

The definitions for the bending and the magnification parameters are

\[
X = \frac{r_1 + r_2}{r_2 - r_1}, \quad M = \frac{u_1 + u_2}{u_2 - u_1}
\]

The magnification parameters of the lenses are \( M_1 = +1 \) \( M_2 = -1 \)

The bending parameters of the lenses are \( X_1 = +1 \) \( X_2 = -1 \)

ABCD-matrix of the system:

\[
M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f_2 & 1 & 0 & 0 \\ 0 & -1/f_1 & 1 & 0 \\ 0 & 0 & -1/z/f_1 & 1 \\ 0 & 0 & -1/z/f_2 & 1 \\ 0 & 0 & -1/z/f_2 & 1 \\ 0 & 0 & -1/z/f_2 & 1 \\ 0 & 0 & -1/z/f_2 & 1 \\ \end{pmatrix}
\]

In the case of perfect length we get

\[
z = f_1 + f_2, \quad \Gamma_o = -\frac{f_1}{f_2}, \quad M = \begin{pmatrix} 1/\Gamma_o & f_1 + f_2 \\ 0 & \Gamma_o \end{pmatrix}
\]
in the case of a defocus $\Delta z$ we have

$$z = \Delta z + f_1 + f_2$$

$$M = \begin{pmatrix}
\frac{1}{\Gamma_o} - \frac{\Delta z}{f_1} & f_1 + f_2 + \Delta z \\
\Delta z & \Gamma_o - \frac{\Delta z}{f_2}
\end{pmatrix}$$

The magnification of an afocal system is defined by the ratio of the field angles as

$$\Gamma = \frac{w'}{w}$$

With the help of the Lagrange invariant, which reads in this case

$$n \cdot x \cdot w = n' \cdot x' \cdot w'$$

we get the magnification by the marginal ray heights

$$\Gamma = \frac{w'}{w} = \frac{x}{x'}$$

Due to the definition of the ABCD matrix

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = M \begin{pmatrix} x \\ u \end{pmatrix}$$

$$x' = M_{11}x + M_{12}u$$

change then is with $u=0$

$$\Gamma = \frac{x}{x'} = M_{11}^{-1} = \frac{1}{1 - \frac{\Gamma_o}{f_1}}$$

$$\Delta \Gamma = \Gamma - \Gamma_o \approx \frac{\Gamma_o^2 \Delta z}{f_1} = \frac{f_1 \Delta z}{f_2^2} = 0.025$$

and the residual angle deviation is obtained with $u=0$ as

$$u' = M_{21} \cdot x + M_{22} \cdot u = \frac{x \cdot \Delta z}{f_1 f_2} = 0.0005$$
Exercise 2-5: Ball Lens

Derive the focal length of a ball lens. What is especially the formula of the focal length for the refractive index $n = 1.5$?
If the ball lens is used symmetrical for an object in a finite distance, what is the overall length of the imaging system?
Derive the condition, that must be fulfilled, that an incoming plane wave is focused onto the back vertex point of a ball lens. What is the focal length of the ball lens for this special setup? Where is the principal plane in this case?

Solution:

The focal length of a thick lens in air is given by the expression

$$f = \frac{-nr_2}{(n-1)\cdot[n(r_1-r_2)-(n-1)d]}$$

With the special data $r_2 = -r$, $r_1 = r$ und $d = 2r$ we get the formula

$$f = \frac{nr}{2(n-1)} = \frac{3}{2}r$$

The last expression is valid for $n=1.5$.

For a symmetrical imaging setup we have the length $L$ as

$$180 = u + i' + (180 - i) \quad , \quad u = i - i'$$

$$y = r \cdot \sin i = r \cdot \frac{\sin u}{n} = z_1 \cdot \tan u \approx z_1 \cdot \sin u = z_1 \cdot (\cos i' \sin i - \sin i' \cos i) \approx z_1 \cdot \left( \sin i - \frac{\sin i'}{n} \right)$$

$$z_1 = \frac{r}{n-1} \quad , \quad z_2 = r \cdot \cos i' \approx r \quad , \quad L = 2 \cdot z_1 + 2 \cdot z_2 = r \cdot \frac{2n}{n-1}$$

With $1/s=0$, $s'=2r$, $n=1$ and $n'=n$, the imaging formula for a single refracting surface (in this case the front surface of the ball) becomes

$$\frac{n'}{s'} + \frac{n}{s} = \frac{n'-n}{r}$$

$$n/2 = n - 1 \quad , \quad n = 2$$

independent from the radius $r$. The focal length then reads $f=r$. The focal length is the distance from the principal plane to the image location. Therefore, the principal plane must be located at the center of the ball lens.