Exercise 1: Refraction angle deviation

The angle deviation $\delta$ at a plane surface between two media with refractive indices $n$ and $n'$ can be written as

$$\sin \frac{\delta}{2} = \frac{n'-n}{2n} \cdot \frac{\sin i'}{\cos \frac{i+i'}{2}}$$

1a) Derive this formula.

1b) Derive a formula for $\delta$ as a function of $n$, $n'$ and $I$ alone

1c) Evaluate the paraxial small angle limit of this formula

Solution:

1a) Re-arrangement of the formula:

$$\sin \frac{\delta}{2} = \frac{n'-n}{2n} \cdot \frac{\sin i'}{\cos \frac{i+i'}{2}}$$

$$\sin \frac{\delta}{2} \cdot \cos \frac{i+i'}{2} = \frac{n'-n}{2n} \cdot \sin i'$$

Left side of this equation with $\delta = i - i''$
\[ \sin \frac{\delta}{2} \cdot \cos \frac{i+i'}{2} = \sin \frac{i-i'}{2} \cdot \cos \frac{i+i'}{2} \]
\[ = \left( \sin \frac{i}{2} \cdot \cos \frac{i'}{2} - \sin \frac{i'}{2} \cdot \cos \frac{i}{2} \right) \cdot \left( \cos \frac{i}{2} \cdot \cos \frac{i'}{2} - \sin \frac{i}{2} \cdot \sin \frac{i'}{2} \right) \]
\[ = \sin i \cdot \cos i' + \sin i' \cdot \cos i - \sin \frac{i}{2} \cdot \sin \frac{i'}{2} = \frac{1}{2} \sin i - \frac{1}{2} \sin i' = \frac{1}{2} \sin i - \frac{n}{2n'} \sin i = \frac{n'-n}{2n} \sin i \]

1b) From the drawing we see:
\[ i - \delta = i' \]
\[ \sin(i - \delta) = \sin i' = \frac{n}{n'} \sin i \]
\[ \sin i \cos \delta - \sin \delta \cos i = \frac{n}{n'} \sin i \]
\[ \sin i \cdot \left( \frac{n^2}{n^2} - \cos \delta \right) = -\cos i \sin \delta \]
\[ \sin^2 i \cdot \left( \frac{n^2}{n^2} - 2 \cdot \cos \delta + \cos^2 \delta \right) = \cos^2 i \cdot \left(1 - \cos^2 \delta \right) \]
\[ \cos^2 \delta - 2 \cdot \sin^2 i \cos \delta + \sin^2 i \cdot \left(1 + \frac{n^2}{n^2} \right) - 1 = 0 \]

Quadratic equation for \( \cos i \) has the solution
\[
\cos \delta = \frac{n}{n'} \sin^2 i + \sqrt{1 + \frac{n^2}{n^2} \sin^4 i - \left(1 + \frac{n^2}{n^2} \right) \sin^2 i} \]

1c) For small angles, the \( \sin i \) is a small quantity and is approximated by its argument. The root is expanded as a Taylor expansion in the form
\[ \cos \delta = \frac{n}{n'} \sin^2 i + \sqrt{1 + \frac{n^2}{n^2} \sin^4 i - \left(1 + \frac{n^2}{n^2} \right) \sin^2 i} \approx \frac{n}{n'} i^2 + \sqrt{1 - \left(1 + \frac{n^2}{n^2} \right) \sin^2 i} \]
\[ \approx \frac{n}{n'} i^2 + \left[1 - \frac{1}{2} \left(1 + \frac{n^2}{n^2} \right) \sin^2 i \right] = 1 + i^2 \cdot \left( \frac{n}{n'} - \frac{1}{2} - \frac{n^2}{2n^2} \right) = 1 + i^2 \cdot \frac{2n'n'' - n^2}{2n^2} = 1 + i^2 \cdot \frac{(n' - n)^2}{2n^2} \]

With the expansion of the cos-function
\[ \cos \delta = 1 - \frac{1}{2} \delta^2 \]

We get by comparing the quadratic terms
\[ 1 - \frac{1}{2} \delta^2 = 1 - i^2 \cdot \frac{(n' - n)^2}{2n^2} \]
\[ \delta = i \cdot \frac{n' - n}{n'} = i \left(1 - \frac{n}{n'} \right) \]
\[ \delta = i - i' \]
Exercise 2: Paraxial Imaging at a Surface

Derive the equation for the imaging condition at a single refracting surface with radius R

\[
\frac{n}{s} + \frac{n'}{s'} = \frac{n' - n}{nR}
\]

from the law of refraction. \( s \) and \( s' \) are the intersection lengths of the locations of object and image along the optical axis. Both are considered to be positive here.

Solution:

\( s \) is considered to be positive here.

Paraxial analysis of the triangles:

Triangle QPC: \[
\sin \psi \approx \psi = \frac{y}{R}
\]
Triangle OPQ: \[
\sin \varphi \approx \varphi = \frac{y}{s}
\]
Triangle O'PQ: \[
\sin \varphi' \approx \varphi' = \frac{y}{s'}
\]
Triangle OPC: \[
\varphi + \psi = i
\]
Triangle O'PC: \[
\varphi' = -i'
\]
Paraxial law of refraction: \[
n \cdot i = n' \cdot i'
\]

(6) in (5), \( i' \) eliminated: \[
\varphi' - \psi = -\frac{n}{n'} \cdot i
\]

(4) in (7), \( i \) eliminated: \[
\varphi' - \psi = -\frac{n}{n'} \left( \varphi + \psi \right)
\]

(1), (2), (3) in (8): \( \varphi, \varphi' \) and \( \psi \) eliminated:

\[
\frac{y}{s'} \cdot \frac{y}{R} = -\frac{n}{n'} \left( \frac{y}{s} + \frac{y}{R} \right)
\]

Rearranged, without \( y \):

\[
\frac{n'}{s'} + \frac{n}{s} = \frac{n' - n}{R}
\]
Exercise 3: Perfect Focussing Hyperbolic Surface

Assume an aspherical surface between two media with indices \( n \) and \( n' \) respectively. Compute the exact surface equation under the assumption, that collimated incoming ray bundle is focussed perfectly. Discuss the result and distinguish the cases \( n > n' \) and \( n < n' \).

Solution:

Refraction
\[ n \cdot \sin i = n' \cdot \sin i' \tag{1} \]

Geometry of triangle
\[ \tan u = \frac{y}{f-z} \tag{2} \]

Angle relation
\[ i = u + i' \tag{3} \]

Surface normal
\[ \tan i = -z'(y) \tag{4} \]

Therefore: Elimination of \( i, i', u \)

(3) into (1)
\[ \frac{dz}{dy} = -\tan i = \frac{n' \sin u}{n + n' \cos u} \]

with
\[ \sin u = \frac{y}{\sqrt{y^2 + (f-z)^2}}, \quad \cos u = \frac{f-z}{\sqrt{y^2 + (f-z)^2}} \]

gives
\[ \frac{dz}{dy} = \frac{n'}{n + n' - \frac{f-z}{\sqrt{y^2 + (f-z)^2}}} = \frac{n' y}{n \sqrt{y^2 + (f-z)^2} + n'(f-z)} \]

Integration gives the equation
\[ y^2 = \left( \frac{n^2}{n'^2} - 1 \right) z^2 - 2 f \cdot \left( \frac{n}{n'} - 1 \right) \cdot z \]

which is an hyperboloid for \( z(y) \) in the case of \( n' > n \) and an ellipsoid for \( n' < n \).

Easy derivation with the help of the Fermat principle of constant optical path length: condition:
\[ n \cdot z + n' \sqrt{(f - z)^2 + y^2} = f \cdot n' \]

from this equation we get by rearranging:

\[ (nz - fu')^2 = n'^2 \left[ y^2 + (f - z)^2 \right] \]

which gives the same equation as before.
Exercise 4: Rod Lens

Consider a thick lens with symmetrical radii of curvature. Calculate the critical thickness $d_c$, for which an incoming ray parallel to the optical axis intersects the lens at its middle point. What happens for $d > d_c$ with the focal length of the lens?

Solution:

Intersection length at the first surface

$$\frac{1}{s'} = \frac{n-1}{nR}$$

critical thickness:

$$d_c = 2s = \frac{2nR}{n-1}$$

Focal length:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2d}{nR_1R_2}$$

with $R_2 = R_1$:

$$\frac{1}{f} = (n-1)\frac{2}{R} - \frac{(n-1)^2d}{nR^2}$$

for $d = d_c$:

$$f = \infty$$

For $d > d_c$, the focal length becomes negative due to the internal focal point.
Exercise 5: Lagrange Invariant at a Single Surface

Consider an imaging surface between two media with indices \( n \) and \( n' \) and radius of curvature \( R \). Calculate the refracted marginal ray from an axis point with aperture angle \( u \) and from a finite object point with height \( y \) through the vertex of the surface in paraxial approximation. From the results calculate the property

\[
L = n \cdot y \cdot \sin u
\]

in the image space.

Solution:

Triangles with the chief ray (green, all length are considered to be positive here)

\[
\tan w \approx w = \frac{y}{s}, \quad \tan w' \approx w' = \frac{y'}{s'}
\]

Chief ray refraction in the vertex point

\[
n \cdot i = n' \cdot i'
\]

with

\[
w = i, \quad w' = i'
\]

we get

\[
n \cdot w = n' \cdot w', \quad \frac{n \cdot y}{s} = \frac{n' \cdot y'}{s'}
\]

\[
\frac{s}{s'} = \frac{n \cdot y}{n' \cdot y'}
\]

Triangles with marginal ray (red):

\[
h = s \cdot \tan u \approx s \cdot u = s' \cdot u'
\]

\[
\frac{s}{s'} = \frac{u'}{u}
\]

Elimination of \( s, s' \):

\[
\frac{s}{s'} = \frac{u'}{u} = \frac{ny}{n' \cdot y'}
\]

\[
L = nyu = n' \cdot y' \cdot u'
\]