Exercise Solutions:
Design and Correction of Optical Systems – Part 2

Exercise 2-1: Achromatization

Consider an achromat shown as below with the focal length $f = 100 \text{ mm}$ with a crown glass ($n_1 = 1.5$ and Abbe number $\nu_1 = 70$) for the positive lens in front and a flint glass ($n_2 = 1.5$) for the negative lens. The cemented surface should be plane and the focal length of the crown lens is given as $f_1 = 70 \text{ mm}$.

a) Calculate the focal length of the flint lens and the radii of curvature of the outer surfaces.
b) Calculate the necessary Abbe number of the flint lens to fulfill achromatization condition. Is this a realistic value?

crown

flint

Solution:
The data are: $n_1 = n_2 = 1.5$, $\nu_1 = 70$, $F_1 = 1/f_1 = 0.014286 \text{ mm}^{-1}$; $F = 1/f = 0.01 \text{ mm}^{-1}$

a) Condition for the focal power:

$F = F_1 + F_2$

Therefore focal length of the flint lens

$F_2 = F - F_1 = -0.004286 \text{ mm}^{-1}$; $f_2 = -233.3 \text{ mm}$

With the formula for a plane-convex lens, we get the radii

$r_1 = (n_1 - 1) \cdot f_1 = 35 \text{ mm}$

$r_2 = -(n_2 - 1) \cdot f_2 = 116.7 \text{ mm}$

b) Condition for achromatization

$\frac{F_1}{\nu_1} + \frac{F_2}{\nu_2} = 0$

The Abbe number therefore is

$\nu_2 = -\nu_2 \cdot \frac{F_1}{F_2} = 21$

This value is realistic, although at the very edge of available glasses.
An optical system of the retro focus type has the following principle arrangement of a negative and a positive lens. With this setup a working distance can be obtained, which is larger than the focal length of the system.

The system should be described by the two focal lengths and the distance \( t \) between the two lenses, which should be considered to be thin.

\[ \text{lens 1} \quad \text{f} \quad \text{lens 2} \quad \text{image plane} \]

\[ f_1 \quad f_2 \quad t \quad s' \]

a) Establish the ABCD-matrix description of the system in the paraxial approximation.

b) Calculate the back focal length \( s' \) of the system.

c) Calculate the focal length \( f \) of the system.

d) Show, that the relative stretching factor \( k = s'/f \) between back focal length and focal length is independent on the focal length of the second lens. Visualize this surprising effect graphically.

e) Discuss the possible application of this setup.

**Solution:**

a) The total system ABCD-matrix reads

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} (1-s'/f_2) \cdot (1-t/f_1) - s'/f_2 & s' \cdot (1-s'/f_2) \\ -1/f_1 - (1-t/f_1) / f_2 & 1-t/f_2 \end{pmatrix}
\]

b) The back focal length is given by the condition \( x' = 0 \) for all values of \( x \). Due to the equation \( x' = Ax + Bu \) with \( u = 0 \) we have \( A = 0 \). This condition gives

\[
A = (1-s'/f_2) \cdot (1-t/f_1) - s'/f_2 = 0
\]

or

\[
s' = \frac{f_2 \cdot (f_1 - t)}{f_1 + f_2 - t}
\]

c) The focal length can be calculated by two different approaches.

The easiest way to calculate the focal length is to compare the expression of \( C \) with the general approach

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -1/f & D \end{pmatrix}
\]

This immediately gives
\[ f = \frac{f_2 \cdot f_1}{f_1 + f_2 - t} \]

A second option is to use the definition of \( f \): determine the length \( z \) after the second lens, which makes \( x' = x \) equal. This is obtained for \( A = 1 \) and delivers
\[ z = - \frac{f_2 \cdot t}{f_1 + f_2 - t} \]

The focal length then is given by
\[ f = s' - z = \frac{f_2 \cdot f_1}{f_1 + f_2 - t} \]

d) We get for the ratio
\[ k = \frac{s'}{f} = 1 - \frac{t}{f_1} \]

which is independent on \( f_2 \). The following figure shows, that this is clear by analyzing the red and the violet triangle. The ratio of the ray height at the second lens relative to the incoming ray determines the ratio between \( s' \) and \( f \). This is independent on the ray bending of the second lens.

e) Additional optical components could be inserted behind the setup since the free working distance is longer than the focal length.
Exercise 2-3: Ball Lens

Derive the focal length of a ball lens. What is especially the formula of the focal length for the refractive index \( n = 1.5 \)?

If the ball lens is used symmetrical for an object in a finite distance, what is the overall length of the imaging system?

Derive the condition that must be fulfilled, that an incoming plane wave is focused onto the back vertex point of a ball lens. What is the focal length of the ball lens for this special setup?

Where is the principal plane in this case?

Solution:

The focal length of a thick lens in air is given by the expression

\[
f = \frac{-nr r_2}{(n-1) \cdot \left[ n(r_1 - r_2) - (n-1)d \right]}
\]

With the special data \( r_2 = -r, r_1 = r \) und \( d = 2r \) we get the formula

\[
f = \frac{nr}{2(n-1)} = \frac{3}{2} \cdot r
\]

The last expression is valid for \( n=1.5 \).

For a symmetrical imaging setup we have the length \( L \) as

\[
180 = u + i' + (180 - i)
\]

\[
y = r \cdot \sin i' = r \cdot \frac{\sin u}{n} = z_1 \cdot \tan u = z_1 \cdot (\cos i' \sin i - \sin i' \cos i) \approx z_1 \cdot \left( \sin i - \frac{\sin i}{n} \right)
\]

\[
z_1 = \frac{r}{n-1}, \quad z_2 = r \cdot \cos i' \approx r, \quad L = 2 \cdot z_1 + 2 \cdot z_2 = r \cdot \frac{2n}{n-1}
\]

With \( 1/s=0, s'=2r, n=1 \) and \( n'=n \), the imaging formula for a single refracting surface (in this case the front surface of the ball) becomes

\[
\frac{n'}{s'} + \frac{n}{s} = \frac{n' - n}{r}
\]

\[
n/2 = n - 1, \quad n=2
\]
independent from the radius $r$. The focal length then reads $f=r$. The focal length is the distance from the principal plane to the image location. Therefore, the principal plane must be located at the center of the ball lens.