Imaging and Aberration Theory

Lecture 3: Eikonal
2013-11-07
Herbert Gross
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Fermat Principle

- Fermat principle: the light takes the ray path, which corresponds to the shortest time of arrival
- The realized path is a minimum and therefore the first derivatives vanish
  \[ \delta L = \delta \int_{P_1}^{P_2} n(x, y, z) \, ds = 0 \]
- Several realized ray paths have the same optical path length
  \[ L = \int_{P_1}^{P_2} n \cdot \vec{s} \cdot d\vec{r} = \text{const.} \]
- The principle is valid for smooth and discrete index distributions
- The Fermat principle states, that a modified ray path must have a larger optical path length.
- The realized path is a minimum and therefore the first derivatives to path variables must vanish:
  \[
  \frac{\partial L(u, v)}{\partial u} = 0, \quad \frac{\partial L(u, v)}{\partial v} = 0
  \]
- The phase has a stationary point with minimal optical path length.
Principle of Stationary Phase

- Principle of stationary phase from an illustrative point of view
- Oscillatory parts are cancelled out
- The light delivers constructive interference in those directions, which has stationary phase contributions: the ray direction perpendicular to the wave
- Critical are stationary phase contributions of boundary points
Principle of Stationary Phase

- Principle of stationary phase from wave optical viewpoint

- Diffraction integral as plane wave decomposition with phase and slowly varying amplitude

\[ E(x, y, z) = \iint E(s_x, s_y) \cdot e^{\frac{2\pi i}{\lambda} (x s_x + y s_y + z s_z)} \, ds_x \, ds_y \]

\[ E(x, y, z) = \iint A(s_x, s_y) \cdot e^{\frac{2\pi i}{\lambda} \left[ L(s_x, s_y) s_x + y s_y + z s_z \right]} \, ds_x \, ds_y \]

- Oscillatory contributions cancel out except the point of stationary phase

\[ \frac{d}{ds_x} \left[ L(s_x, s_y) s_x + y s_y + z s_z \right] = 0 \quad , \quad \frac{d}{ds_y} \left[ L(s_x, s_y) s_x + y s_y + z s_z \right] = 0 \]

- This gives the solution

\[ x = -\frac{dL}{ds_x} + \frac{s_x}{s_z} \cdot z \quad , \quad y = -\frac{dL}{ds_y} + \frac{s_y}{s_z} \cdot z \]

- The light propagates along the ray path perpendicular to the phase / wave front
Law of Malus-Dupin

- Law of Malus-Dupin:
  - equivalence of rays and wavefronts
  - both are orthonormal
  - identical information

- Condition:
  No caustic of rays

- Mathematical:
  Rotation of Eikonal vanish
  \[ \text{rot}(n \cdot \vec{s}) = 0 \]

- Optical system:
  Rays and spherical waves orthonormal

\[ L = \text{const} \]

\[ s \text{ rays} \]

\[ \text{image plane} \]

\[ \text{object plane} \]

\[ y_0 \]

\[ y_1 \]

\[ z_0 \]

\[ z_1 \]
Ray-Wave Equivalent

- Rays and waves carry the same information
- Wave surface is perpendicular on the rays
- Wave is purely geometrical and has no diffraction properties
Lagrange and Hamilton Formulation

- The point eikonal or optical path length

\[ L = \int n(\vec{r}) \, d\vec{r} \]

written in coordinate representation
with derivative with respect to $z$ gives
the optical Lagrange formulation

\[ ds = dz \cdot \sqrt{1 + x'^2 + y'^2} \]

\[ L(x, y, x', y', z) = n(x, y, z) \cdot \sqrt{1 + x'^2 + y'^2} \]

- With the definition of the impulse variables

\[ p_x = \frac{\partial L}{\partial x'} = n \cdot \frac{x'}{\sqrt{1 + x'^2 + y'^2}} = n \cdot \frac{dx}{ds} = n \cdot s_x \]

\[ p_y = \frac{\partial L}{\partial y'} = n \cdot \frac{y'}{\sqrt{1 + x'^2 + y'^2}} = n \cdot \frac{dy}{ds} = n \cdot s_y \]

- The equation of motion reads in vectorial notation
and corresponds to the eikonal equation

\[ \frac{d}{ds} \left( n \cdot \frac{d\vec{r}}{ds} \right) = \nabla n \]

- The Hamilton version is given by the Legendre transform

\[ H(x, y, p_x, p_y) = p_x \cdot x' + p_y \cdot y' - L \]

\[ = -\sqrt{n^2 - p_x^2 - p_y^2} \]
Eikonal Formulation of Imaging

- A ray is described in a plane of constant z by 4 variables
  - point x, y
  - direction or angle $s_x, s_y$

- An optical system transmits an initial ray into a ray in the image space

- In the most general mathematical description, 4 functions control this ray transform

$$
\begin{align*}
x' &= L_1(x, y, s_x, s_y) \\
y' &= L_2(x, y, s_x, s_y) \\
s_x' &= L_3(x, y, s_x, s_y) \\
s_y' &= L_4(x, y, s_x, s_y)
\end{align*}
$$

- The Fermat principle restricts this most general approach to one single function, the so called eikonal function L

$$
\left(x', y', s_x', s_y' \right) = L(x, y, s_x, s_y)
$$

- In reality, there are only 4 degrees of freedom: for a pre-given initial ray, the transferred ray is fixed
Eikonal Formulation of Imaging

- There are 4 possible options to formulate the problem concerning the choice of the independent variables

<table>
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<th>image space</th>
<th>eikonal</th>
<th>inventor</th>
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<td>x,y</td>
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</tr>
<tr>
<td>sₓ,sᵧ</td>
<td>x',y'</td>
<td>angle-point</td>
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<td>afocal, infinity object</td>
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- The different eikonal functions can be calculated via a variable transform by a Legendre transform
- The eikonal functions have singularities, which limits their application
  Example: if conjugated planes are considered, x',y' are fully determined by x,y.
  Therefore there are no 4 independent variables and the point eikonal fails
- There is not one description, which is valid and useful for all possible cases
Point Eikonal

- Point Eikonal \( L(x,y,x',y') \):
  Optical path length from point \( P(x,y) \) in object space to point \( P'(x',y') \) in image space

- Total differential of the Hamilton Eikonal for purely transverse directions and \( z = z' = 0 \)

\[
dl_P(x, y, x', y') = n'(s_x' dx' + s_y' dy') - n(s_x dx + s_y dy)
\]

\[
dl_P(\bar{r}, \bar{r}') = n'd\bar{r} \cdot \bar{s} - n\cdot d\bar{r}' \cdot \bar{s}'
\]

- Differential equations

\[
\frac{\partial L_P}{\partial x} = -n s_x \\
\frac{\partial L_P}{\partial y} = -n s_y \\
\frac{\partial L_P}{\partial x'} = n' s_x' \\
\frac{\partial L_P}{\partial y'} = n' s_y'
\]

- Physical interpretation:
  Given point in object and image space:
  Integration of the equations give the corresponding directions

- Not valid in conjugated planes:
  \( x', y' \) does not depend on path \( s, s' \)
Point-Angle Eikonal

- Legendre-transformation of point eikonal:
  point angle eikonal
  \[ L_{PA}(x, y, s'_x, s'_y) = L_p(x, y, x', y') - n'(x's'_x + y's'_y) \]

- Total differential
  \[ dL_{PA}(x, y, s'_x, s'_y) = -n'(x'ds'_x + y'ds'_y) - n'(s_dx + s_dy) \]

- Derivatives:
  \[ \frac{\partial L_{PA}}{\partial x} = -ns_x \quad \frac{\partial L_{PA}}{\partial y} = -ns_y \quad \frac{\partial L_{PA}}{\partial s'_x} = -n'x' \quad \frac{\partial L_{PA}}{\partial s'_y} = -n'y' \]

- \( L_{PA} \) defines a ray with starting point and final direction

- The point angle eikonal can be used for conjugated planes

- Point angle eikonal is not applicable for an afocal system:
  \( s'_x, s'_y \) are independent of \( x, y \)
Point-Angle Eikonal

- Interpretation of the point angle eikonal
- Optical path length from the starting point P until the point Q' in the image space, which corresponds to the perpendicular projection.

\[ s'_x = n \sin(u) \]
Angle Eikonal

- Corresponding Legendre transform gives the pure angle eikonal according to Schwarzschild
  \[ dL_A(s_x, s_y, s'_x, s'_y) = -n'(x' ds_x' + y' ds_y') + n \cdot (x ds_x + y ds_y) \]

- Vectorial formulation
  \[ dL_A(\bar{s}, \bar{s}') = n \cdot (\bar{r} - \bar{a}) \cdot d\bar{s} - n'(\bar{r}' - \bar{a}') \cdot d\bar{s}' \]

- The ray is defined by its directions only

- Differential equations
  \[ \frac{\partial L_A}{\partial s_x} = nx \quad \frac{\partial L_A}{\partial s_y} = ny \quad \frac{\partial L_A}{\partial s'_x} = -n'x' \quad \frac{\partial L_A}{\partial s'_y} = -n'y' \]

- Interpretation:
  optical path length between the feet points \( Q, Q' \) perpendicular to the ray
Reciprocity Relation

- Point angle eikonal
  \[ \frac{\partial L_{AP}}{\partial s_x} = nx \]
  \[ \frac{\partial L_{AP}}{\partial x'} = n's_x \]

- Independence of mixed derivatives
  \[ \frac{\partial^2 L_{AP}}{\partial s_x \partial x} = n \]
  \[ \frac{\partial^2 L_{AP}}{\partial x' \partial s'_x} = n' \]

- Result:
  - reciprocity relation
    \[ n \cdot \partial s_x \partial x = n' \cdot s'_x \partial x' \]
  - relation between lateral and angle magnification
    \[ n \cdot m_A = n' \cdot m \]
Optical Path Length of an Optical System

- A ray from P goes through a system to P'
- Change of initial point P to Q
- Comparison of optical path of P and Q via A:
  Difference: Hamilton eikonal

\[ \delta L = \overline{QQ'} - \overline{PP'} = \overline{QAA'} \overline{Q'} - \overline{PAA'} \overline{P'} = \overline{P' R'} - \overline{PR} \]
\[ = n' \, dr' \cos \theta' - ndr \cos \theta \]
\[ = n' \bar{s}' \cdot d\bar{r}' - n \bar{s} \cdot d\bar{r} \]

- Total differential: spatial point eikonal
- Differential equations

\[ \frac{\partial L_{\text{Ham}}}{\partial x} = -ns_x \quad \frac{\partial L_{\text{Ham}}}{\partial y} = -ns_y \quad \frac{\partial L_{\text{Ham}}}{\partial z} = -ns_z \]
\[ \frac{\partial L_{\text{Ham}}}{\partial x'} = n's_x' \quad \frac{\partial L_{\text{Ham}}}{\partial y'} = n's_y' \quad \frac{\partial L_{\text{Ham}}}{\partial z'} = n's_z' \]

- Change of initial point:
  the change in the final point is fixed by the eikonal
Paraxial Point Eikonal for a Refracting Surface

- Refracted ray by a spherical dielectric interface path difference

\[ L(r) = n \cdot \sqrt{(z - s)^2 + (r - x)^2} + n' \cdot \sqrt{(-z + s')^2 + (r - x')^2} \]

- Paraxial approximation:
  Taylor expansion for small \( x, x', r, z \)

\[ L(r) = \left[ -ns + n's' - \frac{nx^2}{2s} + \frac{n'x^2}{2s'} \right] + r \left[ + \frac{nx}{s} - \frac{n'x'}{s'} \right] + r^2 \cdot \frac{1}{2} \left[ -\frac{n}{s} + \frac{n'}{s'} - \frac{n' - n}{R} \right] \]

- Stationary phase condition

\[ \frac{dL(r)}{dr} = B + 2r \cdot C = 0 \]

- Angles \( u, u' \)

\[ u = \frac{r - x}{-s}, \quad u' = \frac{r - x'}{s'} \]
Paraxial Point Eikonal for a Refracting Surface

- Paraxial imaging conditions

1. Lens makers formula

\[ + \frac{nx}{s} - \frac{n'x'}{s'} = 0, \quad -\frac{n}{s} + \frac{n'}{s'} - \frac{n' - n}{R} = 0 \]

- definition of magnification

\[ m = \frac{x'}{x} = \frac{n \cdot s'}{n' \cdot s} \]
Point-Angle Eikonal and Lens Aberrations

- The point angle eikonal of a lens is suitable to describe the aberrations.

- The derivatives give the deviations:
  1. of the position:
     - Transverse aberrations as deviation from perfect paraxial location

\[\frac{\partial L_{PA}}{\partial s_x'} = -n'x', \quad \frac{\partial L_{PA}}{\partial s_y'} = -n'y'\]

\[x' = m \cdot x + \Delta x' = -\frac{1}{n'} \cdot \frac{\partial L_{PA}}{\partial s_x'}, \quad \Delta x' = -m \cdot x - \frac{1}{n'} \frac{\partial L_{PA}}{\partial s_x'}\]

\[y' = m \cdot y + \Delta y' = -\frac{1}{n'} \frac{\partial L_{PA}}{\partial s_y'}, \quad \Delta y' = -m \cdot y + \frac{1}{n'} \frac{\partial L_{PA}}{\partial s_y'}\]

- The direction:
  - Angle aberrations correspondingly

\[\frac{\partial L_{PA}}{\partial x} = -ns_x, \quad \frac{\partial L_{PA}}{\partial y} = -ns_y\]

- The mixed derivative of the eikonal gives a relationship between the transverse aberration components.

\[dL_{PA}(x, y, s_x', s_y') = -n'(x'ds_x'+y'ds_y') - n \cdot (s_x dx + s_y dy)\]
Angle Eikonal for a Refracting Surface

- **P** point on ray, **A** arbitrary point on axis

- **Real surface**
  \[ z = \frac{x^2 + y^2}{2R} + \frac{(x^2 + y^2)^2}{8R^3} \cdot (1 + b) + ... \]

- **Angle eikonal**
  \[ dL_A(s, s') = n \cdot (\vec{r} - \vec{a}) \cdot d\vec{s} - n' \cdot (\vec{r}' - \vec{a}') \cdot d\vec{s}' \]

- **In coordinate representation**
  \[ dL_A = n \cdot \left[ x \cdot s_x + y \cdot s_y + (z - a) \cdot s_z \right] - n' \cdot \left[ x' \cdot s'_{x'} + y' \cdot s'_{y'} + (z' - a') \cdot s'_{z'} \right] \]

- **with**
  \[ s_z = \sqrt{1 - s_x^2 - s_y^2} \quad , \quad s'_{z'} = \sqrt{1 - s'_{x'}^2 - s'_{y'}^2} \]
Angle Eikonal for a Refracting Surface

- Result for 4th order Taylor approximation

\[ dL_A = - n \cdot a + n' \cdot a' + n \cdot \left[ x \cdot s_x + y \cdot s_y + \frac{x^2 + y^2}{2R} + \frac{a}{2} \cdot (s^2_x + s^2_y) \right] \]

\[- n' \left[ x \cdot s'_x + y \cdot s'_y + \frac{x^2 + y^2}{2R} + \frac{a'}{2} \cdot (s'^2_x + s'^2_y) \right] \]

\[ + n \cdot \left[ \frac{1 + b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s^2_x + s^2_y)}{4R} + \frac{a}{8} \cdot (s^2_x + s^2_y)^2 \right] \]

\[- n' \left[ \frac{1 + b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s'^2_x + s'^2_y)}{4R} + \frac{a'}{8} \cdot (s'^2_x + s'^2_y)^2 \right] \]

- Use of refraction law to eliminate x,y

\[ dL^{(4)}_A = S_1 \cdot u^2 + S_2 \cdot v^2 + S_3 \cdot w^2 + S_4 \cdot u \cdot v + S_5 \cdot u \cdot w + S_6 \cdot v \cdot w \]

with the rotational invariants

\[ u = s^2_x + s^2_y \] , \[ v = s'^2_x + s'^2_y \] , \[ w = s_x s'_x + s_y s'_y \]

and coefficients \( S_j \)
Angle Eikonal for a Refracting Surface

- Coefficients $S_j$

Description of the optical path as a function of:
1. system data: $n$, $n'$, $R$, $b$
2. ray parameter $a$, $a'$

$$S_1 = \frac{a}{8n^3} - \frac{R}{4(n-n')^2} \left[ \frac{1}{n} + \frac{1+b}{2(n'-n)} \right]$$

$$S_2 = -\frac{a'}{8n'^3} - \frac{R}{4(n-n')^2} \left[ -\frac{1}{n'} + \frac{1+b}{2(n'-n)} \right]$$

$$S_3 = -\frac{(1+b)R}{2(n'-n)^3}$$

$$S_4 = -\frac{R}{4(n-n')^2} \left[ \frac{1}{n} - \frac{1}{n'} + \frac{1+b}{n'-n} \right]$$

$$S_5 = \frac{R}{2(n-n')^2} \left[ \frac{1}{n} + \frac{1+b}{n'-n} \right]$$

$$S_6 = \frac{R}{2(n-n')^2} \left[ -\frac{1}{n} + \frac{1+b}{n'-n} \right]$$
Perfect Imaging

- **Eikonal theory:**
  - perturbation method
  - zero order is the paraxial approximation
  - higher order perturbation corresponds to aberrations

- **Perfect imaging as special cases:**
  1. Stigmatic imaging with conic sections finite-finite (ellipsoid), infinite-fine (parabola)
  2. Special refractive index distributions, e.g. Maxwellian fish-eye
  3. For infinitesimal field size and finite aperture cone: aplanatic imaging condition

- **Imperfect lens:** magnification and transverse aberrations

\[ x' = m \cdot x + \Delta x, \quad y' = m \cdot y + \Delta y \]

- **From point angle eikonal differential equations:**

\[ x' = -\frac{1}{n'} \frac{\partial L_{pA}(x, y, s'_x, s'_y)}{\partial s'_x}, \quad y' = -\frac{1}{n'} \frac{\partial L_{pA}(x, y, s'_x, s'_y)}{\partial s'_y} \]
Perfect Imaging

- Perfect imaging as special cases:

Examples

- Ellipsoid
- Aplanatic
- Concentric
- Hyperboloid

Maxwell lens

\[ n(x, y, z) = \frac{2n_{en\nu}}{1 + \frac{x^2 + y^2 + z^2}{R^2}} \]

Mikhaelian lens

\[ n(r) = \frac{n_0}{\cosh \left( \frac{\pi}{2d} \cdot r \right)} \]
What is 'Ideal'?

- The notation 'ideal' imaging is not unique.
- Ideal is in any case the location of the image point.
- The geometrical ray paths can be different for:
  1. paraxial
  2. ideal / linear collineation
  3. aplanatic
- The photometric properties are different due to non-equidistant sampling.
- If a perfect lens is idealized in a software as one surface, there are principal discrepancies in the location of the intersection points.
If for example a small field area and a widespread ray bundle is considered, a perfect imaging is possible.

The eikonal with the expression

\[ \delta L = n's' \cdot d\vec{r}' - n\vec{s} \cdot d\vec{r} \]

can be written for \( \delta L = 0 \) as

\[ n \cdot \vec{s} \cdot d\vec{r} = n' \cdot s' \cdot d\vec{r}' \]

\[ n \cdot dr \cdot \cos \theta = n' \cdot dr' \cdot \cos \theta' \]

\[ n \cdot \cos \theta = n' \cdot \beta \cdot \cos \theta' \]

In the special case of an angle 90° we get with \( \cos(\theta) = \sin(u) \) the Abbe sine condition

\[ m = \frac{n \sin u}{n' \sin u'} \]

with the lateral magnification

\[ m = \frac{d\vec{r}'}{d\vec{r}} \]
Ray-Wave Relationships

- Concrete calculation of wave aberration: addition of discrete optical path lengths (OPL)

- Reference on chief ray and reference sphere (optical path difference)

- Relation to transverse aberrations

- Conversion between longitudinal transverse and wave aberrations

- Scaling of the phase / wave aberration:
  1. Phase angle in radiant
  2. Light path (OPL) in mm
  3. Light path scaled in $\lambda$

\[
l_{OPL} = \int_{OE}^{AP} n \cdot d\vec{r}
\]

\[
\Delta_{OPD}(x, y) = l_{OPL}(x, y) - l_{OPL}(0,0)
\]

\[
\frac{\partial W}{\partial y_p} = - \frac{\Delta y'}{R - W} \approx - \frac{\Delta y'}{R}
\]

\[
\Delta s' = \frac{R}{y_p} \cdot \Delta y' = \frac{\Delta y'}{\sin u'} = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p}
\]

\[
E(x) = A(x) \cdot e^{i \cdot \varphi(x)}
\]

\[
E(x) = A(x) \cdot e^{i \cdot k \Delta_{OPD}(x)}
\]

\[
E(x) = A(x) \cdot e^{2\pi i \cdot W(x)}
\]
Wave Aberration in Optical Systems

- Definition of optical path length in an optical system:
  Reference sphere around the ideal object point through the center of the pupil
- Chief ray serves as reference
  Difference of OPL : optical path difference OPD
- Practical calculation: discrete sampling of the pupil area, real wave surface represented as matrix
Wave Aberration

- Exact relation between wave aberration and ray deviation
- General expression from geometry
- a describes the lateral aberration
- Substitution of angle by scalar product
- Exact relation is quadratic in R
- Approximation for large R

\[
\Delta W = -\frac{x_p}{R} \cdot \Delta x' - \frac{y_p}{R} \cdot \Delta y' - \frac{x_p^2 + y_p^2}{2 \cdot R} \cdot \Delta z'
\]

with

\[
\tilde{a} = \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}
\]

\[
W = W_0 + n' \cdot \frac{[\tilde{s}_r - \tilde{s} \cdot (\tilde{s} \cdot \tilde{s}_r)] \cdot \tilde{a}}{1 + \tilde{s} \cdot \tilde{s}_r} + n' \cdot \frac{a^2 - (\tilde{s} \cdot \tilde{a})^2}{R \cdot (1 + \cos \theta)}
\]

\[
W = W_\infty + n' \cdot \frac{a^2 - (\tilde{s} \cdot \tilde{a})^2}{2R} \cdot \left[1 + \frac{a^2 - (\tilde{s} \cdot \tilde{a})^2}{4R^2}\right]
\]
Conversion Ray - Wave

- Rays and wavefronts are equivalent
  \[ E(x, y) = A(x, y) \cdot e^{2\pi i W(x, y)} \]

- Phase corresponds to ray direction

\[
\vec{s} = \lambda \cdot \left( \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \sqrt{\left( \frac{1}{\lambda} \right)^2 - \left( \frac{\partial W}{\partial x} \right)^2 - \left( \frac{\partial W}{\partial y} \right)^2} \right)
\]

\[
\frac{\partial W}{\partial x} = \frac{1}{\lambda} \cdot s_x, \quad \frac{\partial W}{\partial y} = \frac{1}{\lambda} \cdot s_y
\]

- Amplitude A is described by ray weighting factor g,
  Transform of area element by Jacobian

\[
g = \Delta A^2 \quad \Delta A' = \frac{\Delta A}{\left| \frac{\partial x'}{\partial x} \cdot \frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y} \cdot \frac{\partial y'}{\partial x} \right|}
\]
Conversion Ray - Wave

- Realization in discrete ray and field sampling

\[ x_p = x_Q + W(x_Q) \tan \varphi \]

\[ \Delta A_j = \Delta x \Delta y \]

\[ \frac{PQ}{W(x_Q)} = \frac{1}{\cos \varphi} \]

- Critical and limits of conversion in regions of a caustic,
  Failure of conversion for crossing of rays
Caustics

- Crossing rays
  - caustic surfaces
  - no unique ray direction
  - singular behavior of the eikonal with special solutions

- The singular solutions describes the envelope of the rays

- In the physical viewpoint, here interference takes place and the amplitude is no longer constant

- The caustic surface can be obtained from the eikonal equation by special techniques
**Geometrical Optics Approximation**

- Helmholtz wave equation
  \[\Delta \tilde{E}(\vec{r}) + k^2 \cdot \tilde{E}(\vec{r}) = 0\]

- Split into phase and amplitude
  \[E(\vec{r}) = A(\vec{r}) e^{-ik_o L(\vec{r})}\]

- The phase/optical path is given by
  \[L = n \cdot \vec{s} \cdot \vec{r}\]

- Insertion, separation of real and imaginary part:
  1. equation
  2. equation

- Approximation of geometrical optics
  Gives the Eikonal equation for the description of ray propagation
  or with ray direction \(s\)

\[2\nabla A \cdot \nabla L + A \cdot \nabla^2 L = 0\]

\[k_o^2 \left(n^2(\vec{r}) - |\nabla L|^2 \right) \cdot A + \nabla^2 A = 0\]

\[a \gg \lambda \ , \ \lambda \cdot \nabla x \ll 1 \ , \ \frac{1}{k_o} \cdot \nabla x \ll 1\]

\[n^2(\vec{r}) - |\nabla L|^2 = 0\]

\[n(\vec{r}) \cdot \vec{s} = \nabla L\]

- Violation of the geometrical optical approximation:
  1. large values of \(\nabla E_0(\vec{r})\) edges, diffraction takes place

  2. large values of \(\nabla L(\vec{r})\) focal points, source points with large angles
Eikonal Equation

- Ansatz for Helmholtz wave equation with Eikonal L
  \[ E(\vec{r}) = E_0(\vec{r}) \cdot e^{-ik_oL(\vec{r})} \]
- Limiting case geometrical optic delivers the Eikonal equation:
  \[ \lambda_o \rightarrow 0 \]
  \[ (\nabla L)^2 = n^2(\vec{r}) \]
- \( L \) describes the optical path length
  \( L = \text{const.} \) are the phase fronts of the wave
  \[ L = \int n(\vec{r}) \, d\vec{r} \]
- Application of the Eikonal equation:
  Numerical solution for the raytracing in inhomogeneous media (gradient)
- Complex \( L \): evanescent damped waves
Raytracing in Grin Media

- Ray: in general curved line in media of nonuniform refractive index
- Numerical solution of Eikonal equation by step-based Runge-Kutta algorithm, 4th order expansion, adaptive step size
- Analytical description of grin media by Taylor expansions of the function $n(x,y,z)$

$$n = n_{o,\lambda} + c_1 h + c_2 h^2 + c_3 h^4 + c_4 h^6 + c_5 z + c_7 z^2 + c_8 z^3 + c_9 z^4 + c_{10} x + c_{11} x^2 + c_{12} x^3 + c_{13} y + c_{14} y^2 + c_{15} y^3$$

- Large computational times necessary for high accuracy
Raytracing in Grin Media

- Numerical solution of the eikonal equation in case of nonhomogeneous media

1. step width \( \Delta t \)

2. scaled optical direction \( \vec{T} = n \cdot \vec{s} = \begin{pmatrix} n \cdot s_x \\ n \cdot s_y \\ n \cdot s_z \end{pmatrix} \)

3. new position and direction \( \vec{r}_{j+1} = \vec{r}_j + \Delta t \left( \vec{T}_j + \frac{\vec{A} + 2\vec{B}}{6} \right) \)

\( \vec{T}_{j+1} = \vec{T}_j + \frac{\vec{A} + 4\vec{B} + \vec{C}}{6} \)

4. Runge-Kutta parameters 4th order

\( \vec{A} = \Delta t \cdot \vec{D}(\vec{r}_j) \)

\( \vec{B} = \Delta t \cdot \vec{D}\left(\vec{r}_j + \frac{\Delta t \cdot \vec{T}_j}{2} + \frac{\Delta t \cdot \vec{A}}{8}\right) \)

\( \vec{C} = \Delta t \cdot \vec{D}\left(\vec{r}_j + \Delta t \cdot \vec{T}_j + \frac{\Delta t \cdot \vec{B}}{2}\right) \)
Gradient Lenses

- Refocusing in parabolic profile
- Helical ray path in 3 dimensions